Journal of Differential Equations **160**, 357–388 (2000)
doi:10.1006/jdeq.1999.3678, available online at http://www.idealibrary.com on IDEAL®

The Free Boundary of a Thermal Wave in a Strongly Absorbing Medium

G. S. Weiss[1]

Tokyo Institute of Technology, O-okayama 2-12-1, Meguro-ku, Tokyo-to, 152 Japan

Received July 6, 1998; revised April 13, 1999; accepted Mars 12, 1999

1. INTRODUCTION

In dimension $n \geqslant 2$ we obtain regularity of the free boundary $\partial\{u > 0\}$ of

$$\partial_t u - \Delta u = -\frac{1+\gamma}{2}\, u^\gamma,\ \gamma \in (0, 1). \tag{1}$$

Our approach is motivated by methods in Liapunov's stability theory and by results concerning the Plateau problem.

Equation (1) has been used in L. K. Martinson [15] and in Ph. Rosenau, S. Kamin [17] to describe the transport of thermal energy in plasma. Alternatively it has been derived as the asymptotic limit of a system proposed by C. Bandle and I. Stakgold in [3] as a simple model for a reaction diffusion process. *Concerning the case of one space dimension* the solution's behaviour near *extinction points* has been extensively studied (see for example A. Friedman, M. A. Herrero [9], M. A. Herrero, J. J. L. Velazquez [12, 13]) and for initial data with compact support the number of extinction points has been estimated (see [9] and X.-Y. Chen, H. Matano, M. Mimura [6]). The paper [6] contains furthermore time-continuity of the set $\{u > 0\} \subset (0, \infty) \times \mathbf{R}$ with respect to Hausdorff distance and it tells us that the one-dimensional free boundary is a subset of a locally finite union of graphs of continuous functions.

For the *stationary* problem H. W. Alt and D. Phillips proved in [1] regularity of the free boundary in higher dimensions. Regarding the time-dependent problem in higher dimesions, regularity of the solution, estimates of the Hausdorff measure of the free boundary, the asymptotic behaviour near *horizontal free boundary points* (points at which the

[1] Partially supported by a Grant-in-Aid for Scientific Research, Ministry of Education, Japan.

0022-0396/00 $35.00

behaviour in time is dominant, see Proposition 4.2) and other results are contained in the paper [5] by H. J. Choe and the author.

This article contains a regularity result for $\partial\{u>0\}$ in higher dimensions: suppose that u is a non-negative solution of the Cauchy problem and that the initial data u^0 satisfy $u^0 \in C_0^{2,\sigma}(\mathbf{R}^n)$ and $(u^0)^{-\gamma}\Delta u^0 \in L^\infty(\mathbf{R}^n)$: then $\partial\{u>0\}$ can be decomposed into a regular part $R = \{(t, x) \in ((0, \infty) \times \mathbf{R}^n) \cap \partial\{u>0\}$: at least one blow-up limit of u at (t, x) is a half-plane solution $\}$ such that $\partial\{u>0\}$ is locally in an open neighborhood of R a $\mathbf{C}^{1/2, 1+\mu}$-surface, and a singular part Σ which is ignored by spatial integration by parts in $\{u>0\}$, i.e.

$$\int_{\{u(t)>0\}} \partial_i \zeta = \int_{\partial_{\mathrm{red}}\{u(t)>0\}} \zeta \nu_i \, d\mathscr{H}^{n-1} = \int_{R \cap \{s=t\}} \zeta \nu_i \, d\mathscr{H}^{n-1}$$

for a.e. $t \in (0, \infty)$ and every $\zeta \in C_0^{0,1}(\mathbf{R}^n)$ (Corollary 8.1); here the reduced boundary $\partial_{\mathrm{red}}\{u(t)>0\}$ is the set of free boundary points at which the outer normal of H. Federer [7, 4.5.5] exists. Let us first remark that while Σ is in the just mentioned sense a set of less relevance, it is in general not a set of small measure: the steady-state solution $((1-\gamma)/2\ |x_1|)^{2/(1-\gamma)}$ satisfies $R = \emptyset, \Sigma = \partial\{u>0\}$ and $\mathscr{H}^{n-1}(\Sigma)>0$. Even worse, when perturbing the stationary equation to $\Delta u = g u^\gamma$ where g is a strictly positive C^∞-function, we expect (in analogy to the counter-example by D. Schaeffer in [18, 2.9] for the case $\gamma = 0$) the appearance of free boundaries such that the *relative boundary* of R is a set of positive $n-1$-dimensional Hausdorff measure. This complicated behaviour of steady-state free boundaries distinguishes our equation from other equations like e.g. the porous medium equation. And we *have* to include the stationary behaviour fully into our considerations for two reasons: first, in the more physical context of [3] the reactant would be replenished at the boundary of some domain and thus be close to a non-trivial steady-state solution for large time. The second and more compelling reason is that by Proposition 4.2 the behaviour close to *each* non-horizontal free boundary point is that of a steady-state solution.

Let us furthermore point out that our situation is different from that of the one-phase Stefan problem (see A. Friedman, D. Kinderlehrer [10] and L. A. Caffarelli [4]) where it was physically justified to assume the temperature to be non-decreasing in time: in the setting of [3] we expect the formation of dead cores in finite time, so there have to be regions where the concentration u is decreasing. Regarding the propagation of a thermal pulse ([17]), the support of a smooth pulse with sufficiently steep slopes will first expand and later on shrink, so there are sign changes of $\partial_t u$.

We do not know whether it is possible to obtain our result by an extension of the sophisticated methods in [1]. We chose here a different approach which is—as already mentioned—related to the concept of Liapunov stability and to results concerning the Plateau problem.

In Section 3 we prove an "epiperimetric inequality" for the class of *half-plane solutions* $H=\{x\mapsto((1-\gamma)/2\max(x\cdot\nu,0))^{2/(1-\gamma)}:\nu\in\partial B_1(0)\}$ and the *boundary-adjusted energy*

$$M(v)=\int_{B_1(0)}(|\nabla v|^2+\max(v,0)^{1+\gamma})-\frac{2}{1-\gamma}\int_{\partial B_1(0)}v^2\,d\mathcal{H}^{n-1}:$$

if c is any non-negative homogeneous function of degree $2/(1-\gamma)$ that is close enough to the some $h\in H$, then there exists a function v with the same boundary values on $\partial B_1(0)$ but with a lower energy value

$$M(v)\leqslant(1-\kappa)\,M(c)+\kappa M(h)\quad\text{(Theorem 3.1).}\tag{2}$$

In hommage to the inequality derived by E. R. Reifenberg for the perimeter, we call (2) by abuse of name "epiperimetric inequality." Our proof however owes nothing to the proof of the epiperimetric inequality in E. R. Reifenberg [16] or that in J. E. Taylor [19] as it works *completely* by indirect methods.

The boundary-adjusted energy plays here the role of the Liapunov function, i.e. its scaled version satisfies a monotonicity formula: defining

$$\begin{aligned}\Phi_{(t_0,x_0)}(r)=r^{-n-2(1+\gamma)/(1-\gamma)}&\int_{B_r(x_0)}(|\nabla u(t_0,\cdot)|^2+\max(u(t_0,\cdot),0)^{1+\gamma})\\&-\frac{2}{1-\gamma}r^{-n+1-4/(1-\gamma)}\int_{\partial B_r(x_0)}u(t_0,\cdot)^2\,d\mathcal{H}^{n-1},\end{aligned}$$

the function $r\mapsto\bar{C}r^{\beta}+\Phi_{(t_0,x_0)}(r)$ is non-decreasing for any $(t_0,x_0)\in\partial\{u>0\}$ at which $|\partial_t(u^{1-\gamma})|$ is Hölder-continuous (Theorem 5.1).

The epiperimetric inequality (2) leads now to the differential inequality

$$\begin{aligned}&\max(\Phi_{(t_0,x_0)}(r)-\Phi_{(t_0,x_0)}(0+),\,C_2r^{\beta})'\\&\quad\geqslant A\,\frac{1}{r}\max(\Phi_{(t_0,x_0)}(r)-\Phi_{(t_0,x_0)}(0+),\,C_2r^{\beta})\end{aligned}$$

which in turn implies Hölder-continuity of $r\mapsto\Phi_{(t_0,x_0)}(r)$ and a convergence estimate for $u(t_0,x_0+r\cdot)/r^{2/(1-\gamma)}$ to the *unique blow-up limit* u_0 (Theorem 6.1).

This reminds very much of the use of Liapunov functions in the theory of linearized stability and of *Liapunov's direct approach* (compare e.g. to Theorem 18.7 and Remark 18.9 in H. Amann [2]). The convergence result itself on the other hand is reminiscent of a result by J. K. Hale and P. Massatt for differentiable gradient systems, by which one obtains single-point ω-limit sets in the case that the multiplicity of the eigenvalue 0 at critical points is 1 ([11, Theorem 4.3]). Let us however point out that our method also works for the obstacle problem where the second variation of the energy vanishes in more than one direction and that *our energy M is not of class* $C^{1,1}$, so a linearization regardless of the direction is not possible. This also means that we cannot apply the center manifold theorem and test the local center manifold for coincidence with the invariant manifold H in order to obtain our result.

In Sections 7 and 8 we derive—mainly by topological methods—the relative openness and $\mathbf{C}^{1/2,\,1+\mu}$-regularity of the set R.

Let us conclude with the remark that it should be possible to obtain higher regularity of R.

2. NOTATION

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$, $B_r(x_0)$ will denote the open n-dimensional ball of center x_0, radius r and volume $r^n\omega_n$, $B'_r(0)$ the open $n-1$-dimensional ball of center 0 and radius r, and e_i the i-th unit vector in $\mathbf{R}^n$. With the exception of Proposition 8.1 the dimension $n \geqslant 2$ and the exponent $\gamma \in (0, 1)$ will remain fixed numbers throughout this paper. We define $Q_r(t_0, x_0) := (t_0 - r^2, t_0 + r^2) \times B_r(x_0)$ to be the cylinder of radius r and height $2r^2$, $Q_r^-(t_0, x_0) := (t_0 - r^2, t_0) \times B_r(x_0)$ its "negative part" and $Q'_r(0, 0) := \{(t, x') \in \mathbf{R}^n : -r^2 < t < r^2 \text{ and } |x'| < r\}$. We shall use the cone $\mathbf{Z}^m_+ := \{(\mu_1, ..., \mu_m) : \mu_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant m\}$ with the norm $|(\mu_1, ..., \mu_m)|_1 := \sum_{i=1}^m \mu_i$. Let the degree $\deg p$ of a polynomial $p(x) = \sum_\mu a_\mu x^\mu$ be defined by $\deg p := \max\{|\mu|_1 : a_\mu \neq 0\}$. We adopt the usual convention $0 \cdot \infty = 0$. Given a set $A \subset \mathbf{R}^n$, we denote its interior by A° and its characteristic function by χ_A. In the text we use the n-dimensional Lebesgue-measure $\mathscr{L}^n$ and the m-dimensional Hausdorff-measure $\mathscr{H}^m$. When considering the boundary of a given set, ν will typically denote the topological outward normal to the boundary and $\nabla_\theta f := \nabla f - \nabla f \cdot \nu\nu$ the surface derivative of a given function f. We shall often use abbreviations for inverse images like $\{u > 0\} := \{x \in \Omega : u(x) > 0\}$, $\{x_n > 0\} := \{x \in \mathbf{R}^n : x_n > 0\}$, $\{s = t\} := \{(s, y) \in \mathbf{R}^{n+1} : s = t\}$ etc. and occasionally we employ the decomposition $x = (x', x_n)$ of a vector $x \in \mathbf{R}^n$. The space $H^{1,2}(B_1(0))$ will be equipped with the inner product $(v, w) := \int_{B_1(0)} (vw + \nabla v \cdot \nabla w)$. Finally, $W_p^{1,2} := \mathbf{W}_p^{2,1}$ and $\mathbf{C}^{\beta,\mu} := \mathbf{H}^{\mu,\beta}$ denote the parabolic Sobolev- and Hölder-spaces as defined in O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva [14].

3. THE EPIPERIMETRIC INEQUALITY

The epiperimetric inequality is an analytic theorem giving information on the asymptotic behaviour of the energy near certain solutions. In order to stress its independence and importance we chose to put it at the beginning of this thesis, however the reader who likes to know more about the context first has the possibility to begin with Sections 4 and 5 which are in turn completely independent of this section.

Our proof of the epiperimetric inequality is fully indirect and allows thereby some insight into the relation between energy decay and algebraic properties of the second variation of the boundary-adjusted energy.

Let us introduce the class $H := \{x \mapsto ((1-\gamma)/2 \max(x \cdot \nu, 0))^{2/(1-\gamma)} : \nu \in \partial B_1(0)\}$ of all half-plane solutions as well as the *boundary-adjusted energy*

$$M(v) := \int_{B_1(0)} (|\nabla v|^2 + \max(v, 0)^{1+\gamma}) - \frac{2}{1-\gamma} \int_{\partial B_1(0)} v^2 \, d\mathscr{H}^{n-1}$$

for $v \in H^{1,2}(B_1(0))$.

Then M takes for $h \in H$ the value $M(h) = \frac{1}{2} M(((1-\gamma)/2\, |x \cdot \nu|)^{2/(1-\gamma)})$

$$= \frac{1}{2} \int_{B_1(0)} \left(\frac{1-\gamma}{2} |x_1|\right)^{2/(1-\gamma)} \left(\frac{1-\gamma}{2} |x_1|\right)^{2\gamma/(1-\gamma)} \left(1 - \frac{1+\gamma}{2}\right)$$

$$= \frac{1}{2} \frac{1-\gamma}{2} \int_{B_1(0)} \left(\frac{1-\gamma}{2} |x_1|\right)^{2(1+\gamma)/(1-\gamma)} =: \frac{\alpha_n}{2} > 0.$$

THEOREM 3.1 (The epiperimetric inequality). *There exist $\kappa \in (0, 1)$ and $\delta \in (0, 1)$ such that the following holds for each non-negative function c in $H^{1,2}(B_1(0))$ that is homogeneous of degree $2/(1-\gamma)$: if $\|c - h\|_{H^{1,2}(B_1(0))} \leqslant \delta$ for some $h \in H$ then there exists a function $v \in H^{1,2}(B_1(0))$ such that $v = c$ on $\partial B_1(0)$ and $M(v) \leqslant (1-\kappa) M(c) + \kappa(\alpha_n/2)$.*

Proof. We suppose towards a contradiction that the epiperimetric inequality does not hold: then there exist sequences of positive reals $\kappa_m \to 0$ and $\delta_m \to 0$ as $m \to \infty$ and there exists a non-negative $c_m \in H^{1,2}(B_1(0))$ that is homogeneous of degree $2/(1-\gamma)$ and satisfies

$$\left\| c_m - \left(\frac{1-\gamma}{2} \max(x_n, 0)\right)^{2/(1-\gamma)} \right\|_{H^{1,2}(B_1(0))} = \inf_{h \in H} \|c_m - h\|_{H^{1,2}(B_1(0))} = \delta_m$$

and $M(v) > (1-\kappa_m) M(c_m) + \kappa_m(\alpha_n/2)$ for every $v \in H^{1,2}(B_1(0))$ with c_m-boundary data on $\partial B_1(0)$. Let us denote $((1-\gamma)/2 \max(x_n, 0))^{2/(1-\gamma)} =: k(x)$.

Subtracting from the inequality the value $M(k)$ we obtain that

$$(1-\kappa_m)(M(c_m) - M(k)) < M(v) - M(k)$$
$$\text{for every } v \in H^{1,2}(B_1(0)) \text{ with } c_m\text{-boundary data on } \partial B_1(0). \qquad (3)$$

Now observe that $(\delta M(k))(\phi)$

$$:= \int_{B_1(0)} (2\nabla k \cdot \nabla\phi + (1+\gamma)\max(k,0)^\gamma \phi) - \frac{2}{1-\gamma}\int_{\partial B_1(0)} 2k\phi \, d\mathcal{H}^{n-1}$$
$$= \int_{\partial B_1(0)} 2\phi\left(\nabla k \cdot \nu - \frac{2}{1-\gamma} k\right) d\mathcal{H}^{n-1} = 0$$

for every $\phi \in H^{1,2}(B_1(0))$.

Thus we are allowed to subtract $(1-\kappa_m)(\delta M(k))(c_m - k)$ from the left-hand side and $(\delta M(k))(v-k)$ from the right-hand side of (3) to obtain that

$$(1-\kappa_m)\Bigg[\int_{B_1(0)} |\nabla(c_m - k)|^2 - \frac{2}{1-\gamma}\int_{\partial B_1(0)} (c_m - k)^2 \, d\mathcal{H}^{n-1}$$
$$+ \int_{B_1(0)\cap\{k=0\}} c_m^{1+\gamma}$$
$$+ \int_{B_1(0)\cap\{k>0\}} \left(\int_0^1\int_0^t (1+\gamma)\gamma(sc_m + (1-s)k)^{\gamma-1}\, ds\, dt\right)(c_m - k)^2\Bigg]$$
$$< \int_{B_1(0)} |\nabla(v-k)|^2 - \frac{2}{1-\gamma}\int_{\partial B_1(0)} (v-k)^2 \, d\mathcal{H}^{n-1}$$
$$+ \int_{B_1(0)} \Big[\chi_{\{v<0\}}(\gamma k^{1+\gamma} - (1+\gamma) v k^\gamma) + \chi_{\{k=0\}} \max(v,0)^{1+\gamma}$$
$$+ \chi_{\{k>0 \wedge v\geqslant 0\}}\left(\int_0^1\int_0^t (1+\gamma)\gamma(sv + (1-s)k)^{\gamma-1}\, ds\, dt\right)(v-k)^2\Big]$$
$$\text{for every } v \in H^{1,2}(B_1(0)) \text{ with } c_m\text{-boundary data on } \partial B_1(0). \qquad (4)$$

Let us introduce the normalized functions $w_m := (c_m - k)/\delta_m$ and choose a subsequence $m \to \infty$ such that $w_m \rightharpoonup w$ weakly in $H^{1,2}(B_1(0))$.

The proof proceeds then in the following four steps:

Step 1. $w = 0$ in $B_1(0) \cap \{x_n < 0\}$.

Step 2. w solves the equation $\Delta w = ((1+\gamma)\,\gamma/2)\,k^{\gamma-1} w$ in $B_1(0) \cap \{k > 0\}$.

Step 3. $w \equiv 0$.

Step 4. $w_m \to 0$ strongly in $H^{1,2}(B_1(0))$ as the subsequence $m \to \infty$.

Step 1.

The sequence

$$\chi_{\{k=0\}} \frac{c_m^{1+\gamma}}{\delta_m^2} + \chi_{\{k>0\}} \left(\int_0^1 \int_0^t (1+\gamma)\,\gamma(sc_m + (1-s)\,k)^{\gamma-1}\,ds\,dt \right) w_m^2$$

is bounded in $L^1(B_1(0))$: we insert $v := (1-\eta)\,c_m + \eta k$ into (4) where $\eta \in H_0^{1,\infty}(B_1(0))$ with values in $(0, 1]$ and obtain that

$$\begin{aligned}
&(1-\kappa_m) \left[\int_{B_1(0) \cap \{k=0\}} \frac{c_m^{1+\gamma}}{\delta_m^2} \right.\\
&\qquad \left. + \int_{B_1(0) \cap \{k>0\}} \left(\int_0^1 \int_0^t (1+\gamma)\,\gamma(sc_m + (1-s)\,k)^{\gamma-1}\,ds\,dt \right) w_m^2 \right]\\
&\quad \leqslant C + \int_{B_1(0) \cap \{k=0\}} (1-\eta)^{1+\gamma} \frac{c_m^{1+\gamma}}{\delta_m^2}\\
&\qquad + \frac{1}{\delta_m^2} \int_{B_1(0) \cap \{k>0\}} [((1-\eta)\,c_m + \eta k)^{1+\gamma}\\
&\qquad - k^{1+\gamma} - (1+\gamma)\,k^\gamma (1-\eta)(c_m - k)]\\
&\quad \leqslant C + \int_{B_1(0) \cap \{k=0\}} (1-\eta)^{1+\gamma} \frac{c_m^{1+\gamma}}{\delta_m^2}\\
&\qquad + \frac{1}{\delta_m^2} \int_{B_1(0) \cap \{k>0\}} [(1-\eta)\,c_m^{1+\gamma} + \eta k^{1+\gamma}\\
&\qquad - k^{1+\gamma} - (1+\gamma)\,k^\gamma (1-\eta)(c_m - k)]\\
&\quad = C + \int_{B_1(0) \cap \{k=0\}} (1-\eta)^{1+\gamma} \frac{c_m^{1+\gamma}}{\delta_m^2}\\
&\qquad + \int_{B_1(0) \cap \{k>0\}} (1-\eta)\,w_m^2 \left(\int_0^1 \int_0^t (1+\gamma)\,\gamma(sc_m + (1-s)\,k)^{\gamma-1}\,ds\,dt \right).
\end{aligned}$$

Choosing $\eta(x)=\tilde\eta(|x|)$ and using the homogeneity of c_m and k this implies that

$$\left(\int_0^1 [(1-\kappa_m)-(1-\tilde\eta(s))^{1+\gamma}]\, s^{n-1+2(1+\gamma)/(1-\gamma)}\, ds\right)$$
$$\times\int_{\partial B_1(0)\cap\{k=0\}} \frac{c_m^{1+\gamma}}{\delta_m^2}\, d\mathscr{H}^{n-1}$$
$$+\left(\int_0^1 (\tilde\eta(s)-\kappa_m)\, s^{n-1+2(1+\gamma)/(1-\gamma)}\, ds\right)$$
$$\times\int_{\partial B_1(0)\cap\{k>0\}} \left(\int_0^1\int_0^\tau (1+\gamma)\,\gamma(\sigma c_m+(1-\sigma)\,k)^{\gamma-1}\, d\sigma\, d\tau\right)$$
$$\times w_m^2\, d\mathscr{H}^{n-1}\leqslant C_1$$

and thereby the indicated boundedness in $L^1(B_1(0))$.

Consequently $w=0$ a.e. in $B_1(0)\cap\{x_n<0\}$ and $w=0$ $\mathscr{H}^{n-1}$-a.e. on $B_1(0)\cap\{x_n=0\}$.

Step 2. Let us now show that w solves the equation $\Delta w=((1+\gamma)\,\gamma/2)\,k^{\gamma-1}w$ weakly in $B_1(0)\cap\{k>0\}$: to this end, we insert $v:=\eta(k+\delta_m\phi)+(1-\eta)\,c_m$ into (4) where $\eta\in C_0^\infty(B_1(0)\cap\{k>0\})$ with values in $[0,1]$ and $\phi\in L^\infty(B_1(0))\cap H^{1,2}(B_1(0))$. Hereupon, (4) becomes for large m of the subsequence

$$\int_{B_1(0)} |\nabla w_m|^2+\int_{B_1(0)\cap\operatorname{supp}\eta} w_m^2\left(\int_0^1\int_0^t(1+\gamma)\,\gamma(sc_m+(1-s)\,k)^{\gamma-1}\, ds\, dt\right)$$
$$< C\kappa_m+\int_{B_1(0)} [|\nabla(\eta\phi)|^2+|\nabla((1-\eta)\,w_m)|^2+2\nabla(\eta\phi)\cdot\nabla((1-\eta)\,w_m)]$$
$$+\int_{B_1(0)\cap\operatorname{supp}\eta}\left(\int_0^1\int_0^t (1+\gamma)\,\gamma(sv+(1-s)\,k)^{\gamma-1}\, ds\, dt\right)$$
$$\times(\eta\phi+(1-\eta)\,w_m)^2.$$

It follows that

$$\int_{B_1(0)} (1-(1-\eta)^2)\,|\nabla w_m|^2$$
$$+\int_{B_1(0)\cap\operatorname{supp}\eta}\left(\int_0^1\int_0^t (1+\gamma)\,\gamma(sc_m+(1-s)\,k)^{\gamma-1}\, ds\, dt\right) w_m^2$$

$$
\begin{aligned}
&\leqslant C\kappa_m + \int_{B_1(0)} [\,|\nabla(\eta\phi)|^2 + |\nabla\eta|^2\, w_m^2 \\
&\quad - 2w_m(1-\eta)\,\nabla\eta\cdot\nabla w_m + 2\nabla(\eta\phi)\cdot\nabla((1-\eta)\,w_m)] \\
&\quad + \int_{B_1(0)\cap\operatorname{supp}\eta} \left(\int_0^1\int_0^t (1+\gamma)\,\gamma(sv+(1-s)\,k)^{\gamma-1}\,ds\,dt\right) \\
&\quad \times(\eta\phi+(1-\eta)\,w_m)^2
\end{aligned}
$$

and that when passing to the limit $m\to\infty$,

$$
\begin{aligned}
&\int_{B_1(0)} |\nabla w|^2 + \int_{B_1(0)\cap\operatorname{supp}\eta} \frac{(1+\gamma)\,\gamma}{2}\,k^{\gamma-1}w^2 \\
&\qquad \leqslant \int_{B_1(0)} |\nabla(\eta\phi+(1-\eta)\,w)|^2 \\
&\qquad\quad + \int_{B_1(0)\cap\operatorname{supp}\eta} \frac{(1+\gamma)\,\gamma}{2}\,k^{\gamma-1}(\eta\phi+(1-\eta)\,w)^2
\end{aligned}
$$

for η and ϕ as above. By approximation we can drop the condition $\phi\in L^\infty(B_1(0))$, and considering any open ball $B\subset\subset B_1(0)\cap\{k>0\}$ we may choose $\eta:=1$ in B and $\phi:=w$ outside B to obtain that

$$
\int_B |\nabla w|^2 + \int_B \frac{(1+\gamma)\,\gamma}{2}\,k^{\gamma-1}w^2 \leqslant \int_B |\nabla\phi|^2 + \int_B \frac{(1+\gamma)\,\gamma}{2}\,k^{\gamma-1}\phi^2
$$

for $\phi\in H^{1,2}(B)$ with w-boundary data on ∂B.

Step 3. We show that $w\equiv 0$. Extending w to a homogeneous function of degree $2/(1-\gamma)$ in $\{x_n>0\}$ and defining

$$
\tilde{k}(x', x_n) := \begin{cases} k(x', x_n), & x_n>0\\ k(x', -x_n), & x_n<0\end{cases}
$$

and

$$
\tilde{w}(x', x_n) := \begin{cases} w(x', x_n), & x_n>0\\ -w(x', -x_n), & x_n<0\end{cases}
$$

we see that $\tilde{w}$ is a homogeneous weak solution of degree $2/(1-\gamma)$ of the equation $\Delta\tilde{w}=((1+\gamma)\,\gamma/2)\,\tilde{k}^{\gamma-1}\tilde{w}$ in $\mathbf{R}^n$. If we consider now a multiindex $\mu\in\mathbf{Z}_+^{n-1}$ and the higher order partial derivative $\partial^\mu\tilde{w}=:\zeta$, then ζ satisfies again the equation $\Delta\zeta=((1+\gamma)\,\gamma/2)\,\tilde{k}^{\gamma-1}\zeta$ in $\mathbf{R}^n$, ζ is by repeated local energy estimates contained in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ and ζ is a homogeneous function of degree $2/(1-\gamma)-|\mu|_1$. From the integrability and homogeneity we infer that $\partial^\mu\tilde{w}\equiv 0$ for $2/(1-\gamma)-|\mu|_1-1\leqslant -n/2$. Thus $x'\mapsto\tilde{w}(x', x_n)$ is a polynomial and the homogeneity and integrability imply the existence of a polynomial p of degree $\deg p < 2/(1-\gamma) + \frac{1}{2} - 1$ such that $w(x', x_n) = x_n^{2/(1-\gamma)} p(x'/x_n)$ for $x_n > 0$. Next we take $\mu \in \mathbf{Z}_+^{n-1}$ such that $|\mu|_1 = \deg p$ and infer from the equation $\Delta \partial^\mu w = (1+\gamma)\,\gamma/2(2/(1-\gamma))^2\, x_n{}^{-2} \partial^\mu w$ in $\{x_n > 0\}$ that $\partial^\mu p\, \Delta(x_n^{2/(1-\gamma)-|\mu|_1}) = (1+\gamma)\,\gamma/2(2/(1-\gamma))^2 \;\; \partial^\mu p\, x_n^{2/(1-\gamma)-|\mu|_1-2}$ in $\{x_n > 0\}$. In the case that the constant $\partial^\mu p \neq 0$ this implies that $(2/(1-\gamma) - |\mu|_1)(2/(1-\gamma) - |\mu|_1 - 1) = (1+\gamma)\,\gamma/2\,(2/(1-\gamma))^2$ and we conclude that either $|\mu|_1 = 1$ or $|\mu|_1 = 2(1+\gamma)/(1-\gamma)$. The latter case can be excluded since $\deg p \leqslant 2/(1-\gamma)$, and in the former case $w(x) = x_n^{2/(1-\gamma)}(d + \ell \cdot x'/x_n)$, whereupon the equation for w yields that

$$\begin{aligned}
&\frac{(1+\gamma)\,\gamma}{2} \left(\frac{2}{1-\gamma}\right)^2 x_n^{2\gamma/(1+\gamma)} \left(d + \ell \cdot \frac{x'}{x_n}\right) \\
&\quad = \frac{2}{1-\gamma}\,\frac{1+\gamma}{1-\gamma}\, x_n^{2\gamma/(1+\gamma)} \left(d + \ell \cdot \frac{x'}{x_n}\right) \\
&\qquad + 2x_n^{2\gamma/(1+\gamma)} \ell \cdot \frac{x'}{x_n} - \frac{4}{1-\gamma}\, x_n^{2\gamma/(1+\gamma)} \ell \cdot \frac{x'}{x_n}.
\end{aligned}$$

We deduce that $d = 0$ and that $w = x_n{}^{1+\gamma/(1-\gamma)} \ell \cdot x'$ in $\{x_n > 0\}$. On the other hand, the information that k is the best approximation in H implies that $(w_m, h-k) \leqslant 1/(2\delta_m)\, \|h-k\|^2_{H^{1,2}(B_1(0))}$ for every $h = ((1-\gamma)/2 \max(x \cdot \nu, 0))^{2/(1-\gamma)} \in H$. Consequently, $o(1) \geqslant$

$$\begin{aligned}
&\int_{B_1(0)} w_m \left(\frac{1-\gamma}{2} \max(x_n, 0)\right)^{(1+\gamma)/(1-\gamma)} x \cdot \xi \\
&\quad + \nabla w_m \cdot \Bigg[\left(\frac{1-\gamma}{2} \max(x_n, 0)\right)^{(1+\gamma)/(1-\gamma)} \xi \\
&\qquad + \frac{1+\gamma}{2} \left(\frac{1-\gamma}{2} \max(x_n, 0)\right)^{2\gamma/(1-\gamma)} x \cdot \xi e_n \Bigg]
\end{aligned}$$

as $\partial B_1(0) - \{e_n\} \ni \nu \to e_n$ and $(\nu - e_n)/(|\nu - e_n|) \to \xi$. Choosing $\xi := \ell$ and passing to the limit in m we obtain that $0 \geqslant$

$$\begin{aligned}
&\int_{B_1(0)} \Bigg[\left(\frac{1-\gamma}{2}\right)^{(1+\gamma)/(1-\gamma)} \max(x_n, 0)^{2(1+\gamma)/(1-\gamma)} (x' \cdot \ell)^2 \\
&\quad + \left(\frac{1-\gamma}{2}\right)^{(1+\gamma)/(1-\gamma)} \max(x_n, 0)^{2(1+\gamma)/(1-\gamma)}\, |\ell|^2 \\
&\quad + \frac{(1+\gamma)^2}{2(1-\gamma)} \left(\frac{1-\gamma}{2}\right)^{2\gamma/(1-\gamma)} \max(x_n, 0)^{4\gamma/(1-\gamma)}\, (x' \cdot \ell)^2 \Bigg].
\end{aligned}$$

Hence $\ell = 0$ and $w \equiv 0$.

Step 4. In order to derive a contradiction to the definition of w_m which tells us that $\|w_m\|_{H^{1,2}(B_1(0))}=1$, it is now sufficient to show the strong convergence of ∇w_m in $L^2(B_1(0))$ as the subsequence $m\to\infty$:

to this end, we choose $v:=(1-\eta)\,c_m+\eta k$ as test function in (4) where $\eta\in H_0^{1,\infty}(B_1(0))$ with values in $[0,1]$ and obtain as in Step 1 that

$$\begin{aligned}
&\int_{B_1(0)}|\nabla w_m|^2+\int_{B_1(0)\cap\{k=0\}}\frac{c_m^{1+\gamma}}{\delta_m^2}\\
&\qquad+\int_{B_1(0)\cap\{k>0\}}\left(\int_0^1\int_0^t(1+\gamma)\,\gamma(sc_m+(1-s)\,k)^{\gamma-1}\,ds\,dt\right)w_m^2\\
&\quad\leqslant C\kappa_m+\int_{B_1(0)\cap\{k=0\}}(1-\eta)^{1+\gamma}\,\frac{c_m^{1+\gamma}}{\delta_m^2}\\
&\quad+\int_{B_1(0)\cap\{k>0\}}(1-\eta)\,w_m^2\left(\int_0^1\int_0^t(1+\gamma)\,\gamma(sc_m+(1-s)\,k)^{\gamma-1}\,ds\,dt\right)\\
&\qquad+\int_{B_1(0)}\left[|\nabla\eta|^2\,w_m^2+(1-\eta)^2\,|\nabla w_m|^2-2w_m(1-\eta)\,\nabla\eta\cdot\nabla w_m\right].
\end{aligned}\tag{5}$$

Now let $\eta(x):=\max(0,\min(1,2(1-|x|)))$, whereupon (5) yields that

$$\int_{B_{1/2}(0)}|\nabla w_m|^2\leqslant C\kappa_m+\int_{B_1(0)}\left[|\nabla\eta|^2\,w_m^2-2w_m(1-\eta)\,\nabla\eta\cdot\nabla w_m\right].$$

At this point the homogeneity of w_m allows us to calculate

$$\begin{aligned}
\int_{B_{1/2}(0)}|\nabla w_m|^2&=\left(\left[\int_0^{1/2}s^{n-1+2(1+\gamma)/(1-\gamma)}\,ds\right)\int_{\partial B_1(0)}|\nabla w_m|^2\,d\mathscr{H}^{n-1}\\
&=\frac{1}{n+\dfrac{2(1-\gamma)}{1-\gamma}}\left(\frac{1}{2}\right)^{n+2(1+\gamma)/(1-\gamma)}\int_{\partial B_1(0)}|\nabla w_m|^2\,d\mathscr{H}^{n-1}\\
&=\left(\frac{1}{2}\right)^{n+2(1+\gamma)/(1-\gamma)}\int_{B_1(0)}|\nabla w_m|^2,
\end{aligned}$$

and we conclude our proof with the estimate $\int_{B_1(0)}|\nabla w_m|^2$

$$\leqslant 2^{n+2(1+\gamma)/(1-\gamma)}\left(C\kappa_m+\int_{B_1(0)}\left[|\nabla\eta|^2\,w_m^2-2w_m(1-\eta)\,\nabla\eta\cdot\nabla w_m\right]\right)\to 0$$

as the subsequence $m\to\infty$.

4. INTRODUCTION TO THE PROBLEM AND PROPERTIES OF SOLUTIONS

Throughout this paper we assume that $\gamma \in (0,1)$, that $\Omega \subset \mathbf{R}^n$ is a bounded domain with $C^{2,\sigma}$ boundary for some $\sigma \in (0,1)$, that the boundary data g are non-negative and satisfy $g \in W^{1,2}_\infty((0,T)\times\Omega)$ for $T<\infty$ and that $u \in L^2((0,T); H^{1,2}(\Omega))$ for $T<\infty$ is a solution of

$$u = g \text{ on } (\{0\}\times\Omega)\cup((0,\infty)\times\partial\Omega)$$

$$\text{and } \partial_t u - \Delta u = -\frac{1+\gamma}{2}\max(u,0)^\gamma \text{ in } (0,\infty)\times\Omega \tag{6}$$

in the sense of distributions.

It is well known that (6) has a unique solution which is by the comparison principle bounded in $(0,T)\times\Omega$ for $T<\infty$. Taking $\min(u,0)$ as test function in the weak equation (6) we see immediately that u has to be non-negative. Standard energy estimates furthermore imply that $u \in H^{1,2}((0,T)\times\Omega)$ for $T<\infty$ and parabolic L^p-theory yields that $u \in W^{1,2}_p((0,T)\times\Omega)$ for $T<\infty$ and $1\leqslant p<\infty$.

Let us also remark that $g(t,x)=0$ for $t\geqslant T_0$ leads by comparison to $\max(0,(1-\gamma)(1+\gamma)/2(2(\sup u(T_0,\cdot))^{1-\gamma}/((1+\gamma)(1-\gamma))-t+T_0))^{1/(1-\gamma)}$ (in $\{t>T_0\}\times\Omega$) to the conclusion that $u(t,x)=0$ for $t\geqslant T_0+2(\sup u(T_0,\cdot)^{1-\gamma}/((1+\gamma)(1-\gamma))$.

The following Theorem 4.1, Proposition 4.1, Corollary 4.1, Lemma 4.1, Proposition 4.2 and Corollary 4.2 have been stated and proved as Theorem 3.1, Proposition 4.1, Corollary 4.2, Lemma 5.1, Proposition 5.2 and Corollary 5.3 in [5]. Note however that the equation considered in [5] does not contain the factor $(1+\gamma)/2$, so several of the statements differ from those in [5] by a factor.

THEOREM 4.1 (Regularity). *There exists a constant $\bar{C}<\infty$ depending only on n,γ,T,M and $\delta\in(0,1)$ such that for each solution u of* (6) *with respect to n,γ and $g\in\mathbf{C}^{(2+\sigma)/2,\,2+\sigma}([0,T]\times\bar{\Omega})$ satisfying*

$$1+\sup_{(0,T)\times\Omega} g + \sup_{\{0\}\times\Omega}|g^{-\gamma}\Delta g| + \sup_{(0,T)\times\partial\Omega}|g^{-\gamma}\partial_t g| \leqslant M<\infty$$

the estimates

$$\|\nabla(u^{(1-\gamma)/2})\|_{L^\infty((\delta,T-\delta)\times\Omega_\delta)} + \|\partial_t(u^{1-\gamma})\|_{L^\infty((0,T)\times\Omega)} \leqslant \bar{C}$$

$$\text{and } \|u^{(1-\gamma)/2}\|_{\mathbf{C}^{1/2,1}((\delta,T-\delta)\times\Omega_\delta)} \leqslant \bar{C}$$

hold (here $\Omega_\delta := \Omega - B_\delta(\partial\Omega)$).

PROPOSITION 4.1 (Non-degeneracy). *There exists a constant* $c>0$ *depending only on* n *and* γ *such that the solution* u *of* (6) *satisfies for every* $(t_0, x_0) \in \overline{\{u>0\}}$ *and every* $Q_r(t_0, x_0) \subset (0, \infty) \times \Omega$ *the estimate*

$$\sup_{Q_r^-(t_0, x_0)} u \geqslant c\, r^{2/(1-\gamma)}.$$

COROLLARY 4.1 (Finite propagation speed of $\{u>0\}$). *There exists a constant* $1 \leqslant S < \infty$ *depending only on* n, γ *and* M *such that for each solution* u *of* (6) *with respect to* n, γ *and* $g \in \mathbf{C}^{(2+\sigma)/2,\, 2+\sigma}([0, \infty) \times \overline{\Omega})$ *satisfying*

$$1 + \sup_{\{0\} \times \Omega} |g^{-\gamma} \Delta g| + \sup_{(0, \infty) \times \partial\Omega} |g^{-\gamma} \partial_t g| \leqslant M < \infty$$

and for every $Q_r^+(t_0, x_0) \subset (0, \infty) \times \Omega$ *the implication*

$$u(t_0, \cdot) = 0 \text{ in } B_r(x_0) \Rightarrow u(t_0 + s^2, \cdot) = 0 \text{ in } B_{\max(0,\, r - Ss)}(x_0)$$

holds.

LEMMA 4.1 (Subsolution property). *The function* $w := |\partial_t(u^{1-\gamma})|^{1/(1-\gamma)}$ *is subcaloric in* $(0, \infty) \times \Omega$ *in the sense that any caloric function* $v \in L^\infty(Q_r(t_0, x_0)) \cap C^0(\overline{Q_r(t_0, x_0)} \cap \{u>0\})$ *satisfying* $Q_r(t_0, x_0) \subset (0, \infty) \times \Omega$, $0 \leqslant v$ *a.e. in* $Q_r(t_0, x_0)$ *and* $(1-\gamma)\, |\partial_t u| \leqslant u^\gamma v^{1-\gamma}$ *in* $(\{t_0 - r^2\} \times B_r(x_0)) \cup ((t_0 - r^2, t_0 + r^2) \times \partial B_r(x_0))$ *must be* $\geqslant w$ *a.e. in* $Q_r(t_0, x_0)$.

We define u to be a solution of the *Cauchy problem* if

$$\begin{gathered} u^0 \in C_0^{2,\sigma}(\mathbf{R}^n),\ (u^0)^{-\gamma} \Delta u^0 \in L^\infty(\mathbf{R}^n), \\ u = u^0 \text{ on } \{0\} \times \mathbf{R}^n, \\ u \geqslant 0 \text{ and } \partial_t u - \Delta u = -\frac{1+\gamma}{2} \max(u, 0)^\gamma \text{ in } (0, \infty) \times \mathbf{R}^n \end{gathered} \tag{7}$$

in the sense of distributions.

Note that this u coincides by Corollary 4.1 in $(0, \infty) \times B_Z(0)$ with the solution of (6) with respect to $g(t, x) = u^0(x)$ and $\Omega = B_Z(0)$ provided that Z has been chosen large enough in terms of n, γ and u^0.

For a solution u of the Cauchy problem (7) we know in addition the following from [6]:

PROPOSITION 4.2 (Horizontal and non-horizontal points). *The following dichotomy holds at each free boundary point* $(t_0, x_0) \in ((0, \infty) \times \Omega) \cap \partial\{u>0\}$:

either $\limsup_{\{u>0\}\ni(t,x)\to(t_0,x_0)} |\partial_t(u^{1-\gamma})| = (1-\gamma)(1+\gamma)/2$ *in which case* (t_0, x_0) *is called a* horizontal point *and* $\max(0, (1-\gamma)(1+\gamma)/2\,(-t))^{1/(1-\gamma)}$ *is the unique* blow-up limit *with respect to every* blow-up sequence $u_m(t,x) = \rho_m{}^{-2/(1-\gamma)} u(t_0 + \rho_m^2 t, x_0 + \rho_m x)$ (*where* $\rho_m \to 0$ *as* $m \to \infty$),

or $\limsup_{\{u>0\}\ni(t,x)\to(t_0,x_0)} |\partial_t(u^{1-\gamma})| = 0$ *in which case* (t_0, x_0) *is called a* non-horizontal point *and every blow-up limit is a non-trivial steady-state solution of* (6) *which is homogeneous of degree* $2/(1-\gamma)$.

Throughout this paper we will often use the term *homogeneous solution of degree* $2/(1-\gamma)$. This will always denote a non-negative homogeneous solution of degree $2/(1-\gamma)$ of class C^2 of the equation $\Delta v = (1+\gamma)/2\, v^\gamma$. Moreover, let us denote by Hor the *set of horizontal points.*

COROLLARY 4.2 (Relative openness of non-horizontal points). *The set of non-horizontal free boundary points* $\partial\{u>0\} - \mathrm{Hor}$ *is open relative to* $((0,\infty)\times\mathbf{R}^n) \cap \partial\{u>0\}$.

We conclude this section with the following

LEMMA 4.2. *Let* u *be a solution of the Cauchy problem* (7) *and assume that* $Q_{2r_0}(t_0, x_0) \cap (\mathrm{Hor} \cup \{t<0\}) = \emptyset$ *and that* $R \subset Q_{2r_0}(t_0, x_0) \cap \partial\{u>0\}$ *satisfies* $0 < \sigma \leqslant \mathscr{L}^{n+1}(Q_r(t,x) \cap \{u=0\})/\mathscr{L}^{n+1}(Q_r)$ *for every* $(t,x) \in Q_{r_0}(t_0, x_0) \cap R$ *and every* $r \leqslant r_0$. *Then there exists* $\beta \in (0,1)$ *depending only on* n, γ *and* σ *and there exists* $C < \infty$ *such that* $|\partial_t(u^{1-\gamma})(t,\xi)| \leqslant C|\xi - x|^\beta$ *for* $(t,x) \in Q_{r_0}(t_0, x_0) \cap R$ *and* $\xi \in B_{r_0}(x)$.

Proof. From Lemma 4.1 and Proposition 4.2 we know that $|\partial_t(u^{1-\gamma})|^{1/(1-\gamma)}$ is a continuous subcaloric function in $Q_{2r_0}(t_0, x_0)$. Lemma A4 of [4] yields therefore constants $\kappa \in (0,1)$ and $\tau \in (0,1)$ depending only on σ and n such that

$$\sup_{Q^-_{\kappa r}(t,x)} |\partial_t(u^{1-\gamma})|^{1/(1-\gamma)} \leqslant \tau \sup_{Q^-_r(t,x)} |\partial_t(u^{1-\gamma})|^{1/(1-\gamma)}$$

for $(t,x) \in Q_{r_0}(t_0, x_0) \cap R$ and $r \leqslant r_0/2$. An iteration leads to the indicated Hölder-continuity.

5. THE MONOTONICITY FORMULA

A powerful tool is now given by the monotonicity formula introduced by the author in an elliptic version ([20, Theorem 3.1]) and in a parabolic version ([21, Theorem 3.1]). As we are going to need both versions in the subsequent sections, we give here simplified proofs of the monotonicity formula for the perturbed elliptic case, Theorem 5.1, and of that for the parabolic case, Theorem 5.2. Since Theorem 5.1 has the advantage of being *local* we shall prefer it to Theorem 5.2 in the remaining part of the paper. This way yields (by very minor modifications of the following proofs) also an independent proof of regularity for the stationary boundary value problem.

THEOREM 5.1 (The monotonicity formula). *Suppose that $\beta \in (0,1)$, that $C<\infty$ and that $K \subset\subset ((0,\infty)\times\Omega)\cap\partial\{u>0\}$ satisfies for $(t_0,x_0)\in K$ and $\delta := \frac{1}{2}\inf_K \operatorname{dist}(\cdot,\partial\Omega)$ the estimate $|\partial_t(u^{1-\gamma})(t_0,x)| \leqslant C\,|x-x_0|^\beta$ for $x\in B_\delta(x_0)$. Then there exists $\bar{C}<\infty$ such that for all $0<\rho<\sigma<\delta$ and every $(t_0,x_0)\in K$ the function*

$$\Phi_{(t_0,x_0)}(r) := r^{-n-2(1+\gamma)/(1-\gamma)} \int_{B_r(x_0)} (|\nabla u(t_0,\cdot)|^2 + \max(u(t_0,\cdot),0)^{1+\gamma})$$
$$-\frac{2}{1-\gamma}\, r^{-n+1-4/(1-\gamma)} \int_{\partial B_r(x_0)} u(t_0,\cdot)^2\, d\mathcal{H}^{n-1},$$

defined in $(0,\delta)$, *satisfies* $\Big|\,\Phi_{(t_0,x_0)}(\sigma)-\Phi_{(t_0,x_0)}(\rho)$

$$-\int_\rho^\sigma r^{-n-2(1+\gamma)/(1-\gamma)} \int_{\partial B_r(x_0)} 2\left(\nabla u(t_0,\cdot)\cdot\nu - \frac{2}{1-\gamma}\,\frac{u(t_0,\cdot)}{r}\right)^2 d\mathcal{H}^{n-1}\, dr\Big|$$
$$\leqslant \bar{C}(\sigma^\beta-\rho^\beta).$$

Proof. We introduce the scaled function $v_r(x) := u(t_0,x_0+rx)/r^{2/(1-\gamma)}$, observe that

$$\Phi_{(t_0,x_0)}(r) = \int_{B_1(0)} (|\nabla v_r|^2 + \max(v_r,0)^{1+\gamma}) - \frac{2}{1-\gamma}\int_{\partial B_1(0)} v_r^2\, d\mathcal{H}^{n-1}$$

and calculate

$$\Phi'_{(t_0,x_0)}(r) = \int_{B_1(0)} \Bigg[2\nabla v_r\cdot\nabla\left(\frac{\nabla u(t_0,x_0+rx)\cdot x}{r^{2/(1-\gamma)}} - \frac{2}{1-\gamma}\frac{1}{r}\,\frac{u(t_0,x_0+rx)}{r^{2/(1-\gamma)}}\right)$$
$$+(1+\gamma)\max(v_r,0)^\gamma\left(\frac{\nabla u(t_0,x_0+rx)\cdot x}{r^{2/(1-\gamma)}}\right.$$
$$\left.-\frac{2}{1-\gamma}\,\frac{1}{r}\,\frac{u(t_0,x_0+rx)}{r^{2/(1-\gamma)}}\right)\Bigg]$$
$$-\frac{2}{1-\gamma}\int_{\partial B_1(0)} 2v_r\left(\frac{\nabla u(t_0,x_0+rx)\cdot x}{r^{2/(1-\gamma)}}\right.$$
$$\left.-\frac{2}{1-\gamma}\,\frac{1}{r}\,\frac{u(t_0,x_0+rx)}{r^{2/(1-\gamma)}}\right) d\mathcal{H}^{n-1}$$

$$= \int_{B_1(0)} (-2\Delta v_r + (1+\gamma) \max(v_r, 0)^\gamma) \frac{1}{r} \left(\nabla v_r \cdot x - \frac{2}{1-\gamma} v_r \right)$$

$$+ \frac{2}{r} \int_{\partial B_1(0)} \left(\nabla v_r \cdot x - \frac{2}{1-\gamma} v_r \right)^2 d\mathscr{H}^{n-1}$$

for $r \in (0, \delta)$. By Theorem 4.1 and the assumed Hölder-continuity we obtain therefore that $|\Phi_{(t_0, x_0)}(\sigma) - \Phi_{(t_0, x_0)}(\rho)$

$$- \int_\rho^\sigma r^{-n-2(1+\gamma)/(1-\gamma)} \int_{\partial B_r(x_0)} 2 \left(\nabla u(t_0, \cdot) \cdot \nu - \frac{2}{1-\gamma} \frac{u(t_0, \cdot)}{r} \right)^2 d\mathscr{H}^{n-1} \, dr \Bigg|$$

$$\leqslant C_1 \int_\rho^\sigma \int_{B_1(0)} \left| 2\partial_t u(t_0, x_0 + rx) \, r^{-2\gamma/(1-\gamma)} \frac{1}{r} \right| dr$$

$$\leqslant C_1 \frac{2C}{1-\gamma} \int_\rho^\sigma \int_{B_1(0)} v_r^\gamma r^{\beta - 1} \, dr$$

$$\leqslant \bar{C}(\sigma^\beta - \rho^\beta).$$

Theorem 5.1 has several immediate and important consequences:

PROPOSITION 5.1. *Let the assumptions of Theorem* 5.1 *be satisfied. Then*:

(1) *For each* $(t_0, x_0) \in K$ *the function* $\Phi_{(t_0, x_0)}$ *has a real right limit* $\Phi_{(t_0, x_0)}(0+)$.

(2) *Let* $(t_0, x_0) \in K$ *and* $0 < \rho_m \to 0$ *be a sequence such that the blow-up sequence* $u_m(t, x) := u(t_0 + \rho_m^2 t, x_0 + \rho_m x)/\rho_m^{2/(1-\gamma)}$ *converges a.e. in* $\mathbf{R}^{n+1}$ *to a blow-up limit* u_0. *Then* u_0 *is a non-trivial steady-state homogeneous solution of degree* $2/(1-\gamma)$.

(3) $\Phi_{(t_0, x_0)}(r) \geqslant -\bar{C} r^\beta$ *for every* $(t_0, x_0) \in K$ *and every* $r \in [0, \delta)$.

(4) *The function* $(t_0, x_0) \mapsto \Phi_{(t_0, x_0)}(0+)$ *is upper semicontinuous on K*.

Proof. (1) follows from the fact that $r \mapsto \Phi_{(t_0, x_0)}(r) + \bar{C} r^\beta$ is by Theorem 4.1 and Theorem 5.1 for each $(t_0, x_0) \in K$ a bounded non-decreasing function.

(2) For each $S < \infty$ the sequence $(u_m)_{m=m_0}^\infty$ is by Theorem 4.1 bounded in $W_\infty^{1,2}(Q_S(0, 0))$. From the assumption we know furthermore that $\sup_{\{0\} \times B_S(0)} |\partial_t (u_m{}^{1-\gamma})| \to 0$ as $m \to \infty$, implying by the non-degeneracy Proposition 4.1 and by the uniqueness of non-negative solutions of the

Cauchy problem that $0 \not\equiv u_0$ does not depend on the time variable. Moreover, we conclude from Theorem 5.1 that for all $0 < S_1 < S_2 < \infty$

$$0 \leftarrow \int_{S_1}^{S_2} r^{-n-2(1+\gamma)/(1-\gamma)} \int_{\partial B_r(0)} 2\left(\nabla u_m(0,\cdot)\cdot\nu - \frac{2}{1-\gamma}\,\frac{u_m(0,\cdot)}{r}\right)^2 d\mathscr{H}^{n-1}\,dr$$

as $m \to \infty$ which yields the desired homogeneity of u_0.

(3) Supposing first that $\Phi_{(t_0,x_0)}(0+) < 0$ for some $(t_0, x_0) \in K$, we obtain a sequence of positive reals $\rho_m \to 0$ as $m \to \infty$ and a sequence $u_m(x) = u(t_0 + \rho_m^2 t, x_0 + \rho_m x)/\rho_m^{2/(1-\gamma)} \to u_0(t, x)$ in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^{n+1})$ as $m \to \infty$ such that

$$0 > \lim_{m\to\infty} \Phi_{(t_0,x_0)}(\rho_m) = \int_{B_1(0)} (|\nabla u_0(0,\cdot)|^2 + \max(u_0(0,\cdot), 0)^{1+\gamma})$$

$$-\frac{2}{1-\gamma}\int_{\partial B_1(0)} u_0(0,\cdot)^2\, d\mathscr{H}^{n-1}.$$

From (2) we know at this point that u_0 is a steady-state homogeneous solution of degree $2/(1-\gamma)$ which leads to the contradiction

$$0 > \int_{B_1(0)} (-\Delta u_0 + \max(u_0, 0)^{\gamma})\, u_0 + \int_{\partial B_1(0)} u_0 \left(\nabla u_0\cdot\nu - \frac{2}{1-\gamma}\, u_0\right) d\mathscr{H}^{n-1}$$

$$= \frac{1-\gamma}{2}\int_{B_1(0)} \max(u_0, 0)^{1+\gamma} \geqslant 0.$$

The statement (3) follows then from Theorem 5.1.

(4) For $\varepsilon > 0$ and $(t_0, x_0), (t, x) \in K$ we obtain from Theorem 5.1 that

$$\Phi_{(t,x)}(0+) \leqslant \Phi_{(t,x)}(\rho) + \frac{\varepsilon}{3} \leqslant \Phi_{(t_0,x_0)}(\rho) + \frac{2}{3}\,\varepsilon \leqslant \Phi_{(t_0,x_0)}(0+) + \varepsilon$$

if we choose first ρ and then $|(t, x) - (t_0, x_0)|$ small enough.

THEOREM 5.2. *Assume that u is a solution of the Cauchy problem (7), that $(t_0, x_0) \in (0, \infty) \times \mathbf{R}^n$, that $T_r^-(t_0) = (t_0 - 4r^2, t_0 - r^2) \times \mathbf{R}^n$, that $0 < \rho < \sigma < \sqrt{t_0/2}$ and that*

$$G_{(t_0,x_0)}(t, x) = 4\pi(t_0 - t)\, |4\pi(t_0 - t)|^{-n/2-1} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right)$$

is the backwards heat kernel. Then

$$\Psi^-_{(t_0,x_0)}(r) = r^{-2-2(1+\gamma)/(1-\gamma)} \int_{T_r^-(t_0)} (|\nabla u|^2 + \max(u,0)^{1+\gamma})\, G_{(t_0,x_0)}$$
$$- \frac{2}{1-\gamma}\,\frac{1}{2}\, r^{-4/(1-\gamma)} \int_{T_r^-(t_0)} \frac{1}{t_0 - t}\, u^2 G_{(t_0,x_0)}$$

satisfies the monotonicity formula

$$\Psi^-_{(t_0,x_0)}(\sigma) - \Psi^-_{(t_0,x_0)}(\rho) = \int_\rho^\sigma r^{-1-4/(1-\gamma)} \int_{T_r^-(t_0)} \frac{1}{t_0-t} \Big(\nabla u \cdot (x - x_0)$$
$$- 2(t_0 - t)\,\partial_t u - \frac{2}{1-\gamma}\, u\Big)^2 G_{(t_0,x_0)}\, dr \geq 0.$$

Let $\Psi^-_{(t_0,x_0)}(0+)$ *denote the right limit* $\lim_{r\to 0} \Psi^-_{(t_0,x_0)}(r) \in [-\infty, \infty)$. *Then* $(t_0, x_0) \mapsto \Psi^-_{(t_0,x_0)}(0+)$ *is an upper semicontinuous function in* $(0,\infty)\times \mathbf{R}^n$.

Proof. We introduce the scaled function $u_r(t,x) := u(t_0 + r^2 t,\, x_0 + rx)/r^{2/(1-\gamma)}$, observe that

$$\Psi^-_{(t_0,x_0)}(r) = \int_{T_1^-(0)} (|\nabla u_r|^2 + \max(u_r,0)^{1+\gamma})\, G_{(0,0)}$$
$$- \frac{2}{1-\gamma}\,\frac{1}{2} \int_{T_1^-(0)} \frac{1}{-t}\, u_r{}^2 G_{(0,0)}$$

and calculate $(\Psi^-_{(t_0,x_0)}(r))'$

$$= \int_{T_1^-(0)} \Bigg[2\nabla u_r \cdot \nabla \Bigg(\frac{\nabla u(t_0 + r^2 t,\, x_0 + rx)\cdot x}{r^{2/(1-\gamma)}} + \frac{2rt\,\partial_t u(t_0 + r^2 t,\, x_0 + rx)}{r^{2/(1-\gamma)}}$$
$$- \frac{2}{1-\gamma}\,\frac{1}{r}\,\frac{u(t_0 + r^2 t,\, x_0 + rx)}{r^{2/(1-\gamma)}}\Bigg)$$
$$+ (1+\gamma)\max(u_r,0)^\gamma \Bigg(\frac{\nabla u(t_0 + r^2 t,\, x_0 + rx)\cdot x}{r^{2/(1-\gamma)}} + \frac{2rt\,\partial_t u(t_0 + r^2 t,\, x_0 + rx)}{r^{2/(1-\gamma)}}$$
$$- \frac{2}{1-\gamma}\,\frac{1}{r}\,\frac{u(t_0 + r^2 t,\, x_0 + rx)}{r^{2/(1-\gamma)}}\Bigg)\Bigg]\, G_{(0,0)}$$
$$- \frac{2}{1-\gamma}\,\frac{1}{2} \int_{T_1^-(0)} \frac{1}{-t}\, 2u_r \Bigg(\frac{\nabla u(t_0 + r^2 t,\, x_0 + rx)\cdot x}{r^{2/(1-\gamma)}}$$
$$+ \frac{2rt\,\partial_t u(t_0 + r^2 t,\, x_0 + rx)}{r^{2/(1-\gamma)}} - \frac{2}{1-\gamma}\,\frac{1}{r}\,\frac{u(t_0 + r^2 t,\, x_0 + rx)}{r^{2/(1-\gamma)}}\Bigg)\, G_{(0,0)}$$

$$
\begin{aligned}
&= \int_{T_1^-(0)} \Big[(-2\,\Delta u_r + (1+\gamma)\max(u_r,0)^\gamma + 2\,\partial_t u_r) \\
&\quad \times \frac{1}{r}\left(\nabla u_r\cdot x + 2t\,\partial_t u_r - \frac{2}{1-\gamma}\,u_r\right) \\
&\quad -\frac{2}{r}\,\partial_t u_r\left(\nabla u_r\cdot x + 2t\,\partial_t u_r - \frac{2}{1-\gamma}\,u_r\right)\Big]\,G_{(0,0)} \\
&\quad -2\nabla u_r\cdot\nabla G_{(0,0)}\,\frac{1}{r}\left(\nabla u_r\cdot x + 2t\,\partial_t u_r - \frac{2}{1-\gamma}\,u_r\right) \\
&\quad +\frac{2}{1-\gamma}\int_{T_1^-(0)} \frac{1}{t}\,u_r\,\frac{1}{r}\left(\nabla u_r\cdot x + 2t\,\partial_t u_r - \frac{2}{1-\gamma}\,u_r\right) G_{(0,0)} \\
&= \frac{1}{r}\int_{T_1^-(0)} \left(-2\,\partial_t u_r - \frac{1}{t}\,\nabla u_r\cdot x + \frac{2}{1-\gamma}\,\frac{1}{t}\,u_r\right) \\
&\quad \times\left(\nabla u_r\cdot x + 2t\,\partial_t u_r - \frac{2}{1-\gamma}\,u_r\right) G_{(0,0)} \\
&= r^{-1-4/(1-\gamma)} \int_{T_r^-(t_0)} \frac{1}{t_0-t} \\
&\quad \times\left(\nabla u\cdot(x-x_0) - 2(t_0-t)\,\partial_t u - \frac{2}{1-\gamma}\,u\right)^2 G_{(t_0,x_0)}\,dr
\end{aligned}
$$

for $r\in(0,\sqrt{t_0}/2)$.

For $\varepsilon>0$, $m<\infty$ and $(t,x)\in(0,\infty)\times\mathbf{R}^n$ the inequalities $\Psi^-_{(t,x)}(0+)\leqslant \Psi^-_{(t,x)}(\rho)$

$$
\begin{aligned}
&\leqslant \Psi^-_{(t_0,x_0)}(\rho) + \frac{\varepsilon}{2} \\
&\leqslant \begin{cases} \Psi^-_{(t_0,x_0)}(0+) + \varepsilon \text{ in the case } \Psi^-_{(t_0,x_0)}(0+) > -\infty \\ -m \text{ in the case } \Psi^-_{(t_0,x_0)}(0+) = -\infty \end{cases}
\end{aligned}
$$

hold provided that we choose first ρ and then $|(t,x)-(t_0,x_0)|$ small.

6. AN ENERGY DECAY ESTIMATE AND UNIQUENESS OF BLOW-UP LIMITS

In this section we show that an epiperimetric inequality always implies an energy decay estimate and uniqueness of blow-up limits. More precisely:

THEOREM 6.1 (Energy decay, uniqueness of blow-up limits). *Suppose that* $\beta \in (0,1)$, *that* $C < \infty$ *and that* $K \subset\subset ((0,\infty) \times \Omega) \cap \partial\{u > 0\}$ *satisfies for* $(t_0, x_0) \in K$ *and* $\delta := \frac{1}{2} \inf_K \operatorname{dist}(\cdot, \partial\Omega)$ *the estimate* $|\partial_t (u^{1-\gamma})(t_0, x)| \leqslant C\,|x - x_0|^\beta$ *for* $x \in B_\delta(x_0)$. *Assume furthermore that the epiperimetric inequality holds with* $\kappa \in (0,1)$ *for each* $c_r(x) := (|x|/r)^{2/(1-\gamma)} u(t_0, x_0 + rx/|x|)$ *such that* $r \leqslant r_0 < 1$, *and let* u_0 *denote an arbitrary blow-up limit of* u *at the point* (t_0, x_0).

Then for each $\Lambda \in (0, \min(\beta, (n + 2(1+\gamma)/(1-\gamma))\,\kappa/(1-\kappa)))$ *there exists* $C^* < \infty$ *such that*

$$|\Phi_{(t_0,x_0)}(r) - \Phi_{(t_0,x_0)}(0+)| \leqslant C^* r^\Lambda \textit{ for } (t_0, x_0) \in K \textit{ and } r \in (0, r_0),$$

$$\int_{\partial B_1(0)} \left| \frac{u(t_0, x_0 + rx)}{r^{2/(1-\gamma)}} - u_0(0, x) \right| d\mathcal{H}^{n-1} \leqslant C^* r^{\Lambda/2} \textit{ for } (t_0, x_0) \in K$$

and $r \in (0, r_0/2)$, *and* u_0 *is the unique blow-up limit of* u *at the point* (t_0, x_0).

Proof. We define $e(r) := \Phi_{(t_0,x_0)}(r) - \Phi_{(t_0,x_0)}(0+)$, $v(x) := u(t_0, x)$ and $v_r(x) := u(t_0, x_0 + rx)/r^{2/(1-\gamma)}$ and calculate

$$\begin{aligned}
e'(r) &= r^{-n-\frac{2(1+\gamma)}{1-\gamma}} \int_{\partial B_r(x_0)} (|\nabla v|^2 + \max(v,0)^{1+\gamma})\, d\mathcal{H}^{n-1} \\
&\quad - \frac{2}{1-\gamma}\, r^{-n+1-4/(1-\gamma)} \int_{\partial B_r(x_0)} 2v \nabla v \cdot \nu \, d\mathcal{H}^{n-1} \\
&\quad - \frac{2}{1-\gamma}\, \frac{n-1}{r}\, r^{-n+1-4/(1-\gamma)} \int_{\partial B_r(x_0)} v^2 \, d\mathcal{H}^{n-1} \\
&\quad + \frac{2}{1-\gamma}\, r^{-n-4/(1-\gamma)} \int_{\partial B_r(x_0)} v^2 \, d\mathcal{H}^{n-1} \\
&\quad - \left(n + \frac{2(1+\gamma)}{1-\gamma}\right) \frac{e(r)}{r} - \left(n + \frac{2(1+\gamma)}{1-\gamma}\right) \frac{1}{r}\, \Phi_{(t_0,x_0)}(0+) \\
&= r^{-1} \Bigg[\int_{\partial B_1(0)} \Bigg(|\nabla v_r|^2 + \max(v_r, 0)^{1+\gamma} - \frac{4}{1-\gamma}\, v_r \nabla v_r \cdot \nu + \left(\frac{2}{1-\gamma}\right)^2 v_r^2 \\
&\quad + \left(\frac{2}{1-\gamma}\right)^2 v_r{}^2 - \frac{2}{1-\gamma} \left(\frac{4}{1-\gamma} + n - 2 \right) v_r^2 \Bigg) d\mathcal{H}^{n-1} \\
&\quad - \left(n + \frac{2(1+\gamma)}{1-\gamma}\right) \Phi_{(t_0,x_0)}(0+) \Bigg] - \frac{1}{r} \left(n + \frac{2(1+\gamma)}{1-\gamma}\right) e(r)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant r^{-1}\left[\int_{\partial B_1(0)}\left(|\nabla_\theta v_r|^2+\max(v_r,0)^{1+\gamma}+\left(\frac{2}{1-\gamma}\right)^2 v_r^2\right.\right.\\
&\qquad\left.\left.-\frac{2}{1-\gamma}\left(\frac{4}{1-\gamma}+n-2\right)v_r^2\right)d\mathscr{H}^{n-1}-\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)\Phi_{(t_0,x_0)}(0+)\right]\\
&\qquad-\frac{1}{r}\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)e(r)\\
&=r^{-1}\left[\int_{\partial B_1(0)}(|\nabla c_r|^2+\max(c_r,0)^{1+\gamma})\,d\mathscr{H}^{n-1}\right.\\
&\qquad-\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)\frac{2}{1-\gamma}\\
&\qquad\left.\times\int_{\partial B_1(0)}c_r^2\,d\mathscr{H}^{n-1}-\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)\Phi_{(t_0,x_0)}(0+)\right]\\
&\qquad-\frac{1}{r}\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)e(r)\\
&=\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)\frac{1}{r}\,(M(c_r)-\Phi_{(t_0,x_0)}(0+)-e(r)).
\end{aligned}
$$

At this point we employ the minimality of v_r with respect to the functional

$$
\begin{aligned}
\phi\mapsto&\int_{B_1(0)}(|\nabla\phi|^2+\max(\phi,0)^{1+\gamma})-\frac{2}{1-\gamma}\int_{\partial B_1(0)}\phi^2\,d\mathscr{H}^{n-1}\\
&+\int_{B_1(0)}2\phi\,\partial_t u(t_0,x_0+rx)\,r^{-2\gamma/(1-\gamma)}
\end{aligned}
$$

and fixed boundary values on $\partial B_1(0)$ as well as the assumption that the epiperimetric inequality $M(w)\leqslant(1-\kappa)\,M(c_r)+\kappa\Phi_{(t_0,x_0)}(0+)$ holds for the minimizer w of M with c_r-boundary data on $\partial B_1(0)$, and we obtain by the assumed Hölder-continuity and Theorem 4.1 the estimate

$$
\begin{aligned}
e'(r)\geqslant&\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)\frac{1}{r}\,\frac{1}{1-\kappa}\left(M(v_r)-\Phi_{(t_0,x_0)}(0+)\right.\\
&\left.-2\int_{B_1(0)}(w-v_r)\,\partial_t u(t_0,x_0+rx)\,r^{-2\gamma/(1-\gamma)}\right)\\
&-\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)\frac{1}{r}\,e(r)\\
\geqslant&\left(n+\frac{2(1+\gamma)}{1-\gamma}\right)\frac{1}{r}\,\frac{\kappa}{1-\kappa}\,e(r)-C_1 r^{\beta-1}.
\end{aligned}
$$

Consequently, for each $\Lambda \in (0, \min(\beta, (n+2(1+\gamma)/(1-\gamma))\,\kappa/(1-\kappa)))$ there exists $C_2 < \infty$ such that

$$\max(e(r), C_2 r^\beta)' \geqslant \Lambda \frac{1}{r} \max(e(r), C_2 r^\beta) \text{ for } r \in (0, r_0),$$

$$\log(\max(e(s), C_2 s^\beta))' \geqslant \Lambda \frac{1}{s} \text{ for } s \in (0, r_0),$$

and integrating this inequality from r to r_0 we obtain that

$$\max(e(r), C_2 r^\beta) \leqslant \max(e(r_0), C_2 {r_0}^\beta) \left(\frac{r}{r_0}\right)^\Lambda \text{ for } r \in (0, r_0).$$

Combined with the monotonicity formula Theorem 5.1 this yields the first statement of this theorem.

Using once more the monotonicity formula Theorem 5.1 we get for $0 < \rho < \sigma \leqslant r_0$ an estimate of the form

$$\begin{aligned}
&\int_{\partial B_1(0)} \left| \frac{u(t_0, x_0 + \sigma x)}{\sigma^{2/(1-\gamma)}} - \frac{u(t_0, x_0 + \rho x)}{\rho^{2/(1-\gamma)}} \right| d\mathcal{H}^{n-1} \\
&\leqslant \int_\rho^\sigma r^{1-n-2/(1-\gamma)} \int_{\partial B_r(x_0)} \left| \nabla u(t_0, \cdot) \cdot \nu - \frac{2}{1-\gamma} \frac{u(t_0, \cdot)}{r} \right| d\mathcal{H}^{n-1} dr \\
&\leqslant \sqrt{n\omega_n} \int_\rho^\sigma r^{1-n-2/(1-\gamma)} r^{(n-1)/2} r^{n/2 + (1+\gamma)/(1-\gamma)} \\
&\quad \times \left(r^{-n-2(1+\gamma)/(1-\gamma)} \int_{\partial B_r(x_0)} \left| \nabla u(t_0, \cdot) \cdot \nu - \frac{2}{1-\gamma} \frac{u}{r} \right|^2 d\mathcal{H}^{n-1} \right)^{1/2} dr \\
&\leqslant \sqrt{\frac{n\omega_n}{2}} \int_\rho^\sigma r^{-1/2} (C_3 r^{(\beta-1)/2} + \sqrt{|e'(r)|})\, dr \\
&\leqslant C_4 \sigma^{\beta/2} + \sqrt{\frac{n\omega_n}{2}} (\log(\sigma) - \log(\rho))^{1/2} |e(\sigma) - e(\rho)|^{1/2}.
\end{aligned}$$

Considering now $0 < 2\rho < 2r \leqslant r_0$ and intervals $[2^k, 2^{k+1}) \ni \rho$ and $[2^\ell, 2^{\ell+1}) \ni r$ the already proved part of the theorem yields that

$$\begin{aligned}
&\int_{\partial B_1(0)} \left| \frac{u(t_0, x_0 + r x)}{r^{2/(1-\gamma)}} - \frac{u(t_0, x_0 + \rho x)}{\rho^{2/(1-\gamma)}} \right| d\mathcal{H}^{n-1} \\
&\leqslant C_4 \sum_{i=k}^{\ell} (2^{i+1})^{\beta/2} + C_5 \sum_{i=k}^{\ell} (\log(2^{i+1}) - \log(2^i))^{1/2} |e(2^{i+1}) - e(2^i)|^{1/2}
\end{aligned}$$

$$= C_4 \sum_{i=k}^{\ell} (2^{i+1})^{\beta/2} + C_6 \sum_{i=k}^{\ell} |e(2^{i+1}) - e(2^i)|^{1/2}$$

$$\leqslant C_4 \sum_{j=-\ell-1}^{+\infty} (2^j)^{-\beta/2} + C_7 \sum_{j=-\ell-1}^{+\infty} (2^j)^{-\Lambda/2}$$

$$= C_4 \left(\frac{1}{1-c} - \frac{1-c^{-\ell-1}}{1-c} \right) + C_7 \left(\frac{1}{1-d} - \frac{1-d^{-\ell-1}}{1-d} \right) \Bigg]$$

where $c = 2^{-\beta/2} \in (0, 1)$ and $d = 2^{-\Lambda/2} \in (0, 1)$. Thus

$$\int_{\partial B_1(0)} \left| \frac{u(t_0, x_0 + rx)}{r^{2/(1-\gamma)}} - \frac{u(t_0, x_0 + \rho x)}{\rho^{2/(1-\gamma)}} \right| d\mathscr{H}^{n-1} \leqslant C_8 r^{\Lambda/2}$$

for $r \in (0, r_0/2)$ and $\rho \in (0, r)$, and letting $u(t_0, x_0 + \rho_m \cdot)/\rho_m^{2/(1-\gamma)} \to u_0(0, \cdot)$ as a certain sequence $\rho_m \to 0$ finishes our proof.

7. ASYMPTOTIC BEHAVIOUR NEAR REGULAR POINTS

The aim of this section is to verify the assumptions of Theorem 6.1 uniformly in an open neighborhood of a regular free boundary point, i.e. a free boundary point at which at least one blow-up limit coincides with a half-plane solution.

LEMMA 7.1. *The half-plane solutions are (in the $H^{1,2}(B_1(0))$-metric) isolated within the class of homogeneous solutions of degree* $2/(1-\gamma)$.

Proof. We suppose towards a contradiction that this does not hold: then there exists a sequence of homogeneous solutions of degree $2/(1-\gamma)$, say $(k_m)_{m \in \mathbf{N}}$, such that $0 < \inf_{h \in H} \|k_m - h\|_{H^{1,2}(B_1(0))} = \|k_m - ((1-\gamma)/2 \max(x_n, 0))^{2/(1-\gamma)}\|_{H^{1,2}(B_1(0))} =: \delta_m \to 0$ as $m \to \infty$. When passing to a subsequence $m \to \infty$ such that $(k_m - ((1-\gamma)/2 \max(x_n, 0))^{2/(1-\gamma)})/\delta_m =: w_m \rightharpoonup w$ weakly in $H^{1,2}(B_1(0))$, the limit w is still a homogeneous function of degree $2/(1-\gamma)$. The boundedness of k_m in $H^{2,\infty}(B_1(0))$ as well as the non-degeneracy Proposition 4.1 imply furthermore that w is a solution of $\Delta w = (1+\gamma)\,\gamma/2\ \ ((1-\gamma)/2)^{-2}\, x_n{}^{-2} w$ in $B_1(0) \cap \{x_n > 0\}$ and that $w = 0$ a.e. in $B_1(0) \cap \{x_n < 0\}$.

Moreover we know that $((1-\gamma)/2\ \max(x_n, 0))^{2/(1-\gamma)}$ is the best approximation to k_m among all half-plane solutions. But then it follows exactly as in Step 3 of the proof of the epiperimetric inequality Theorem 3.1 that $w \equiv 0$. In order to obtain a contradiction to the assumption $\delta_m > 0$ by which $\|w_m\|_{H^{1,2}(B_1(0))} = 1$, it is therefore sufficient to show the strong convergence of ∇w_m to ∇w in $L^2(B_1(0))$ as the subsequence $m \to \infty$: by the compact imbedding on the boundary

$$\begin{aligned}\int_{B_1(0)} |\nabla w_m|^2 &= \int_{\partial B_1(0)} w_m \nabla w_m \cdot \nu \, d\mathcal{H}^{n-1} - \int_{B_1(0)} w_m \Delta w_m \\ &= \frac{2}{1-\gamma} \int_{\partial B_1(0)} w_m^2 \, d\mathcal{H}^{n-1} - \frac{1+\gamma}{2} \frac{1}{\delta_m^2} \\ &\quad \times \int_{B_1(0)} \left(k_m - \left(\frac{1-\gamma}{2} \max(x_n, 0) \right)^{2/(1-\gamma)} \right) \\ &\quad \times \left(k_m^\gamma - \left(\frac{1-\gamma}{2} \max(x_n, 0) \right)^{2\gamma/(1-\gamma)} \right) \\ &\leqslant \frac{2}{1-\gamma} \int_{\partial B_1(0)} w_m^2 \, d\mathcal{H}^{n-1} \to 0\end{aligned}$$

as the subsequence $m \to \infty$.

LEMMA 7.2. *Let u be a solution of the Cauchy problem* (7) *and suppose that* $(t_0, x_0) \in R := \{(t, x) \in ((0, \infty) \times \mathbf{R}^n) \cap \partial\{u > 0\}$: *at least one blow-up limit of u at (t, x) is a half-plane solution* $\}$. *Then there exists $\delta > 0$ such that* $\Psi^-_{(t,x)}(0+) \leqslant \Psi^-_{(t_0,x_0)}(0+)$ *for every* $(t, x) \in Q_\delta(t_0, x_0) \cap \partial\{u > 0\}$.

Proof. We assume towards a contradiction that this does not hold. Then there is a sequence $((0, \infty) \times \mathbf{R}^n) \cap \partial\{u > 0\} \ni (t_m, x_m) \to (t_0, x_0)$ such that $\Psi^-_{(t_m,x_m)}(0+) > \Psi^-_{(t_0,x_0)}(0+)$. Consequently the following holds for every sequence $\rho_m \to 0$ as $m \to \infty$:

$$|\Psi^-_{(t_m,x_m)}(\rho_m) - \Psi^-_{(t_m,x_m)}(0+)| + |\Psi^-_{(t_m,x_m)}(0+) - \Psi^-_{(t_0,x_0)}(0+)| \leqslant \varepsilon \quad \text{for } m \geqslant m_0.$$

In case this is not true we find a sequence $\rho_m \to 0$ as $m \to \infty$ such that

$$\begin{aligned}\varepsilon &< \Psi^-_{(t_m,x_m)}(\rho_m) - \Psi^-_{(t_m,x_m)}(0+) + \Psi^-_{(t_m,x_m)}(0+) - \Psi^-_{(t_0,x_0)}(0+) \\ &= \Psi^-_{(t_m,x_m)}(\rho_m) - \Psi^-_{(t_m,x_m)}(\rho) + \Psi^-_{(t_m,x_m)}(\rho) \\ &\quad - \Psi^-_{(t_0,x_0)}(\rho) + \Psi^-_{(t_0,x_0)}(\rho) \\ &\quad - \Psi^-_{(t_0,x_0)}(0+) \leqslant \Psi^-_{(t_m,x_m)}(\rho) - \Psi^-_{(t_0,x_0)}(\rho) + \Psi^-_{(t_0,x_0)}(\rho) - \Psi^-_{(t_0,x_0)}(0+) \\ &< \varepsilon \text{ if we choose first } \rho \text{ small and then } m \text{ large, a contradiction.}\end{aligned}$$

Considering now *any* sequence $u(t_m + \tau_m^2 \cdot, x_m + \tau_m \cdot)/\tau_m^{2/(1-\gamma)}$ such that $\tau_m \to 0$ as $m \to \infty$, the monotonicity formula Theorem 5.2, (8), the fact that $u(t_m + \tau_m^2 \cdot, x_m + \tau_m \cdot)/\tau_m^{2(1-\gamma)}$ is bounded in $H^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^{n+1})$ and Proposition 4.2 imply that every limit u_0 of $u(t_m + \tau_m^2 \cdot, x_m + \tau_m \cdot)/\tau_m^{2/(1-\gamma)}$ must be a steady-state homogeneous solution of degree $2/(1-\gamma)$.

Since each blow-up limit $\tilde{u}_0$ with respect to (t_m, x_m) satisfies by assumption $\Psi^-_{(t_m, x_m)}(0+) > \Psi^-_{(t_0, x_0)}(0+)$ and thus by Lemma 7.1 $\operatorname{dist}_{H^{1,2}(B_1(0))}(\tilde{u}_0, H) \geqslant \delta_1 > 0$, and since at least one blow-up limit $\bar{u}_0$ with respect to $(t_0, x_0) \in R$ is a half-plane solution, we obtain by a continuity argument for each $\theta \in (0, 1)$ a sequence $\tau_m \to 0$ such that $\operatorname{dist}_{H^{1,2}(B_1(0))}(u(t_m + \tau_m^2 \cdot, x_m + \tau_m \cdot)/\tau_m^{2/(1-\gamma)}, H) = \theta\delta_1$ as $m \to \infty$. Passing to the limit in m yields a steady-state homogeneous solution u_0 of degree $2/(1-\gamma)$ satisfying $\operatorname{dist}_{H^{1,2}(B_1(0))}(u_0, H) = \theta\delta_1$, for small θ a contradiction to Lemma 8.

THEOREM 7.1. *Let u be a solution of the Cauchy problem* (7) *and suppose that $(t_0, x_0) \in R$. Then there exist $\delta > 0$, a function v: $\overline{Q_\delta(t_0, x_0)} \cap R \to \partial B_1(0)$ and constants $r_1 > 0$ and $C < \infty$ such that*

$$\int_{\partial B_1(0)} \left| \frac{u(t_1, x_1 + rx)}{r^{2/(1-\gamma)}} - \left(\frac{1-\gamma}{2} \max(x \cdot v(t_1, x_1), 0) \right)^{2/(1-\gamma)} \right| d\mathscr{H}^{n-1} \leqslant C r^{\Lambda/2}$$

for every $(t_1, x_1) \in \overline{Q_\delta(t_0, x_0)} \cap R$ and every $r \leqslant r_1$; here Λ is the exponent of Theorem 8.

Proof. By Corollary 4.2 and Lemma 7.2 there exists $\delta > 0$ such that $\overline{Q_{2\delta}(t_0, x_0)} \cap (\mathrm{Hor} \cup \{t < 0\}) = \varnothing$ and $\Psi^-_{(t,x)}(0+) \leqslant \Psi^-_{(t_0, x_0)}(0+) = \alpha_n/2$ for every $(t, x) \in \overline{Q_{2\delta}(t_0, x_0)} \cap \partial\{u > 0\}$. As in the proof of Lemma 7.2 we obtain therefore that for $\varepsilon > 0$ there exists $\tilde{\delta} > 0$ such that

$$|\Psi^-_{(t,x)}(r) - \Psi^-_{(t,x)}(0+)| \leqslant \varepsilon \text{ for every } (t, x) \in \overline{Q_{2\delta}(t_0, x_0)} \cap R \text{ and every } r \leqslant \tilde{\delta}: \tag{9}$$

in case this is not so we find a sequence $r_m \to 0$ and a sequence $R \cap \overline{Q_{2\delta}(t_0, x_0)} \ni (t_m, x_m) \to (\bar{t}, \bar{x}) \in \overline{Q_{2\delta}(t_0, x_0)} \cap \partial\{u > 0\}$ as $m \to \infty$ such that $\varepsilon <$

$$\begin{aligned}
&\Psi^-_{(t_m, x_m)}(r_m) - \Psi^-_{(\bar{t}, \bar{x})}(0+) + \Psi^-_{(\bar{t}, \bar{x})}(0+) - \frac{\alpha_n}{2} \\
&\quad \leqslant \Psi^-_{(t_m, x_m)}(r_m) - \Psi^-_{(\bar{t}, \bar{x})}(0+) \\
&\quad = \Psi^-_{(t_m, x_m)}(r_m) - \Psi^-_{(t_m, x_m)}(\rho) + \Psi^-_{(t_m, x_m)}(\rho) \\
&\qquad - \Psi^-_{(\bar{t}, \bar{x})}(\rho) + \Psi^-_{(\bar{t}, \bar{x})}(\rho) - \Psi^-_{(\bar{t}, \bar{x})}(0+) \\
&\quad \leqslant \Psi^-_{(t_m, x_m)}(\rho) - \Psi^-_{(\bar{t}, \bar{x})}(\rho) + \Psi^-_{(\bar{t}, \bar{x})}(\rho) - \Psi^-_{(\bar{t}, \bar{x})}(0+) < \varepsilon
\end{aligned}$$

if we choose first ρ small and then m large, a contradiction.

Once more it follows from Theorem 5.2, (9), the regularity of u as well as Proposition 4.2 that every limit of *any* sequence $u(t_m+\rho_m^2\cdot, x_m+\rho_m\cdot)/\rho_m{}^{2/(1-\gamma)}$ such that $(t_m, x_m)\in\overline{Q_{2\delta}(t_0, x_0)}\cap R$ and $\rho_m\to 0$ as $m\to\infty$ must be a steady-state homogeneous solution of degree $2/(1-\gamma)$.

One consequence is that the assumptions of Lemma 4.2 can be verified uniformly in $(\bar t, \bar x)\in\overline{Q_{2\delta}(t_0, x_0)}\cap R$: we maintain that $\mathscr{L}^{n+1}(Q_r(\bar t, \bar x)\cap\{u=0\})/\mathscr{L}^{n+1}(Q_r)\geqslant\frac{1}{4}$ for every $(\bar t, \bar x)\in\overline{Q_{2\delta}(t_0, x_0)}\cap R$ and $r\leqslant\bar r_0$: if this was not the case, we would obtain by Proposition 4.1 sequences $\rho_m\to 0$ and $\overline{Q_{2\delta}(t_0, x_0)}\cap R\ni(t_m, x_m)\to(\tilde t, \tilde x)$ as $m\to\infty$ such that $u(t_m+\rho_m^2\cdot, x_m+\rho_m\cdot)/\rho_m^{2/(1-\gamma)}$ converges to a steady-state homogeneous solution of degree $2/(1-\gamma)$ which is not a half-plane solution. Since then by Lemma 7.1 $\mathrm{dist}_{H^{1,2}(B_1(0))}(u(t_m+\rho_m^2\cdot, x_m+\rho_m\cdot)/\rho_m^{2/(1-\gamma)}, H)\geqslant\sigma>0$ as $m\to\infty$ and since $(t_m, x_m)\in R$ we would find by a continuity argument for each $\theta\in(0, 1)$ a sequence $\tau_m\to 0$ as $m\to\infty$ such that $u(t_m+\tau_m^2\cdot, x_m+\tau_m\cdot)/\tau_m^{2/(1-\gamma)}$ converges to a steady-state homogeneous solution u_0 of degree $2/(1-\gamma)$ satisfying $\mathrm{dist}_{H^{1,2}(B_1(0))}(u_0, H)=\theta\sigma>0$, for small θ a contradiction to Lemma 7.1.

In view of Lemma 4.2, Theorem 6.1 and Theorem 3.1 it is therefore sufficient to show that $\mathrm{dist}_{H^{1,2}(\partial B_1(0))}(u(\bar t, \bar x+r\cdot)/r^{2/(1-\gamma)}, H)<\varepsilon$ for every $(\bar t, \bar x)\in\overline{Q_\delta(t_0, x_0)}\cap R$ and every $r\leqslant 2r_0$. Supposing towards a contradiction that this does not hold, we obtain sequences $\overline{Q_\delta(t_0, x_0)}\cap R\ni(t_m, x_m)\to(\tilde t, \tilde x)$ and $\rho_m\to 0$ as $m\to\infty$ such that $\mathrm{dist}_{H^{1,2}(\partial B_1(0))}(u(t_m, x_m+\rho_m\cdot)/\rho_m^{2/(1-\gamma)}, H)\geqslant\sigma>0$ as $m\to\infty$. Since $(t_m, x_m)\in R$, we obtain by a continuity argument for each $\theta\in(0, 1)$ a sequence $\tau_m\to 0$ as $m\to\infty$ such that $u(t_m+\tau_m^2\cdot, x_m+\tau_m\cdot)/\tau_m^{2/(1-\gamma)}$ satisfies $\mathrm{dist}_{H^{1,2}(\partial B_1(0))}(u(t_m, x_m+\tau_m\cdot)/\tau_m^{2/(1-\gamma)}, H)=\theta\sigma>0$ and converges to a steady-state homogeneous solution u_0 of degree $2/(1-\gamma)$ as $m\to\infty$, for small θ a contradiction to Lemma 7.1.

Making use of the solution's regularity and of Corollary 4.2 we obtain the following

COROLLARY 7.1 (Differentiability). *Let the assumptions of Theorem* 7.1 *be satisfied. Then there exist* $\delta>0$, *a function* v: $\overline{Q_\delta(t_0, x_0)}\cap R\to\partial B_1(0)$ *and* $\omega(z)/z\to 0$ *as* $z\to 0$ *such that*

$$\left|u(t, x)-\left(\frac{1-\gamma}{2}\max((x-\bar x)\cdot v(\bar t, \bar x), 0)\right)^{2/(1-\gamma)}\right|$$

$$\leqslant\omega(|x-\bar x|^{2/(1-\gamma)}+|t-\bar t|^{1/(1-\gamma)})$$

for every $(\bar t, \bar x)\in\overline{Q_\delta(t_0, x_0)}\cap R$.

8. REGULARITY

We show now that the half-plane solutions take a lower energy value than any other homogeneous solution of degree $2/(1-\gamma)$ whose coincidence set has a non-empty interior. Together with Theorem 7.1 this will imply the relative openness and uniform cone-flatness of the set R.

PROPOSITION 8.1. *Let $v \not\equiv 0$ be any homogeneous solution of degree $2/(1-\gamma)$ satisfying $\{v=0\}^\circ \neq \emptyset$. Then $M(v) \geqslant \alpha_n/2$, and equality implies that v is a half-plane solution.*

Proof. In one space dimension the statement is an immediate consequence of the homogeneity.

In order to derive the inequality for higher dimensions we assume inductively that it holds for every dimension $\leqslant n-1$ and that it is violated by a homogeneous solution v of degree $2/(1-\gamma)$ in dimension n, that $\{v=0\}$ contains the ball B and that $\partial B \cap \partial\{v>0\} \supset \{e_n\}$. The upper semicontinuity of the function $x \mapsto \Phi_x(0+)$ (defined with respect to $v(x)$), Proposition 5.1(4) and the homogeneity of v imply now via $\Phi_{e_n}(0+) = \limsup_{m\to\infty} \Phi_{e_n/m}(0+) \leqslant \Phi_0(0+) < \alpha_n/2$ that every blow-up limit v_0 of v at the point e_n satisfies the inequality $M(v_0) < \alpha_n/2$.

Now the homogeneity of v tells us that v_0 must be constant in the direction of the vector e_n and that again $\{v_0=0\}^\circ \neq \emptyset$, so $\tilde{v} := v_0|_{\mathbf{R}^{n-1}}$ is a homogeneous solution of degree $2/(1-\gamma)$ satisfying $\{\tilde{v}=0\}^\circ \neq \emptyset$ and

$$\begin{aligned}
\frac{\alpha_n}{2} &> \int_{B_1(0)} (|\nabla v_0|^2 + \max(v_0, 0)^{1+\gamma}) - \frac{2}{1-\gamma} \int_{\partial B_1(0)} v_0^2 \, d\mathcal{H}^{n-1} \\
&= \frac{1-\gamma}{2} \int_{B_1(0)} \max(v_0, 0)^{1+\gamma} = \frac{1-\gamma}{2} \int_{B_1(0)} w \\
&= (1-\gamma) \int_{\{|x'|<1\}} \int_0^{\sqrt{1-|x'|^2}} dx_n \, \tilde{w}(x') \, dx' \\
&= \left(\int_0^1 \sqrt{1-t^2} t^{n-2+2(1+\gamma)/(1-\gamma)} \, dt \right) (1-\gamma) \int_{\partial B_1'(0)} \tilde{w} \, d\mathcal{H}^{n-2} \\
&= \left(2 \left(n-1+\frac{2(1+\gamma)}{1-\gamma} \right) \int_0^1 \sqrt{1-t^2} t^{n-2+2(1+\gamma)/(1-\gamma)} \, dt \right) \\
&\quad \times \frac{1-\gamma}{2} \int_{B_1'(0)} \max(\tilde{v}, 0)^{1+\gamma}
\end{aligned}$$

$$= \left(2\left(n-1+\frac{2(1+\gamma)}{1-\gamma}\right)\int_0^1 \sqrt{1-t^2}\,t^{n-2+2(1+\gamma)/(1-\gamma)}\,dt\right)$$

$$\times\left(\int_{B_1'(0)}(|\nabla\tilde v|^2+\max(\tilde v,0)^{1+\gamma})-\frac{2}{1-\gamma}\int_{\partial B_1'(0)}\tilde v^2\,d\mathcal{H}^{n-2}\right)$$

(here $w:=\max(v,0)^{1+\gamma}$ and $\tilde w:=\max(\tilde v,0)^{1+\gamma}$). Since the same calculation—when being done for $((1-\gamma)/2\max(x_1,0))^{2/(1-\gamma)}$ instead of v_0—also yields that $\alpha_n/2=(2(n-1+2(1+\gamma)/(1-\gamma))\ \int_0^1\sqrt{1-t^2}\,t^{n-2+2(1+\gamma)/(1-\gamma)}\,dt)$ $\alpha_{n-1}/2$, we obtain a contradiction to the induction hypothesis.

Finally, we assume inductively that the second part of the statement holds for every dimension $\leqslant n-1$ and consider the case of a homogeneous solution v of degree $2/(1-\gamma)$ in dimension n satisfying $M(v)=\alpha_n/2$, $\{v=0\}\supset B$ and $\partial B\cap\partial\{v>0\}\supset\{e_n\}$. As in the first part of the proof we obtain that every blow-up limit v_0 of v at the point e_n satisfies the inequality $M(v_0)\leqslant\alpha_n/2$, that v_0 is constant in the direction of e_n and that $\{v_0=0\}^\circ\neq\emptyset$. Defining again $\tilde v:=v_0|_{\mathbf{R}^{n-1}}$, $\tilde v$ is a homogeneous solution of degree $2/(1-\gamma)$ satisfying $\{\tilde v=0\}^\circ\neq\emptyset$, and the calculation in the first part of the proof yields that

$$\int_{B_1'(0)}(|\nabla\tilde v|^2+\max(\tilde v,0)^{1+\gamma})-\frac{2}{1-\gamma}\int_{\partial B_1'(0)}\tilde v^2\,d\mathcal{H}^{n-2}\leqslant\frac{\alpha_{n-1}}{2}.$$

Then the already proved part of the statement as well as the induction hypothesis imply that $\tilde v$ must be a half-plane solution. For $0<r_m\to 0$ as $m\to\infty$, every blow-up limit of v at the point r_me_n must consequently be a half-plane solution and—assuming that $v\notin H$—we find by a continuity argument for each $\theta\in(0,1)$ a sequence $\rho_m\to 0$ as $m\to\infty$ such that $\mathrm{dist}_{H^{1,2}(B_1(0))}(v(r_me_n+\rho_m\,\cdot\,)/\rho_m^2, H)=\theta\,\mathrm{dist}_{H^{1,2}(B_1(0))}(v,H)>0$. On the other hand, it follows as in the proof of Theorem 7.1 that $v(r_me_n+\rho_m\,\cdot\,)/\rho_m^2$ converges in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ to a homogeneous solution v^* of degree $2/(1-\gamma)$ as a subsequence $m\to\infty$. Then, however, $\mathrm{dist}_{H^{1,2}(B_1(0))}(v^*,H)=\theta\,\mathrm{dist}_{H^{1,2}(B_1(0))}(v,H)>0$ which contradicts for small θ the isolation property Lemma 7.1.

THEOREM 8.1. *Let u be a solution of the Cauchy problem* (7). *Then $\partial\{u>0\}$ is locally in an open neighborhood of the set R a $\mathbf{C}^{1/2,1+\mu}$-surface. The space outward normal $\nu(t,x)$ to $\partial\{u>0\}$ is locally in R a Hölder-continuous function.*

Proof. *Step* 1 (Uniform cone-flatness). Let $(t_0,x_0)\in R$ and let δ be the constant in the statement of Theorem 7.1. Then for each $\epsilon>0$ there exists a $\delta_1>0$ such that

$$u(s, y) = 0 \qquad \text{for} \quad (\bar{t}, \bar{x}) \in \overline{Q_\delta(t_0, x_0)} \cap R$$

$$\text{and } (s, y) \in \overline{Q_{\delta_1}(\bar{t}, \bar{x})} \text{ satisfying } (y - \bar{x}) \cdot \nu(\bar{t}, \bar{x}) < -\varepsilon \max(|y - \bar{x}|, \sqrt{|s - \bar{t}|}),$$

$$\text{and } u(s, y) > 0 \qquad \text{for} \quad (\bar{t}, \bar{x}) \in \overline{Q_\delta(t_0, x_0)} \cap R$$

$$\text{and } (s, y) \in \overline{Q_{\delta_1}(\bar{t}, \bar{x})} \text{ satisfying } (y - \bar{x}) \cdot \nu(\bar{t}, \bar{x}) > \varepsilon \max(|y - \bar{x}|, \sqrt{|s - \bar{t}|}): \tag{10}$$

assuming that (10) does not hold, we obtain a sequence $\overline{Q_\delta(t_0, x_0)} \cap R \ni (t_m, x_m) \to (\tilde{t}, \tilde{x})$ and a sequence $(s_m, y_m)_{m \in \mathbf{N}}$ such that $\rho_m := \max(|y_m - x_m|, \sqrt{|s_m - t_m|}) \to 0$ as $m \to \infty$ and

$$\begin{aligned} &\text{either } u(s_m, y_m) > 0 \quad &&\text{and} \quad &&(y_m - x_m) \cdot \nu(t_m, x_m) < -\varepsilon \rho_m \\ &\text{or } u(s_m, y_m) = 0 \quad &&\text{and} \quad &&(y_m - x_m) \cdot \nu(t_m, x_m) > \varepsilon \rho_m. \end{aligned} \tag{11}$$

Observe now that the estimate of Theorem 7.1 extends by continuity to $(t_1, x_1) \in \overline{Q_\delta(t_0, x_0)} \cap \bar{R}$ and implies (e.g. via uniqueness of $\nu(t_1, x_1)$) that ν is continuous on $\overline{Q_\delta(t_0, x_0)} \cap \bar{R}$. Hence we infer from Theorem 7.1, from the continuity of $|\partial_t(u^{1-\gamma})|$ in $\overline{Q_\delta(t_0, x_0)}$ as well as from the regularity and non-degeneracy of the solution that the sequence $u_m(t, x) := u(t_m + \rho_m^2 t, x_m + \rho_m x)/\rho_m{}^{2/(1-\gamma)}$ converges in $C^{0,\sigma}_{\mathrm{loc}}(\mathbf{R}^{n+1})$ to $((1-\gamma)/2 \max(x \cdot \nu(\tilde{t}, \tilde{x}), 0))^{2/(1-\gamma)}$ as $m \to \infty$ and that $u_m = 0$ on each compact subset K of $\{x \cdot \nu(\tilde{t}, \tilde{x}) < 0\}$ provided that $m \geqslant m(K)$. This, however, contradicts (11) for large m.

Step 2 ($Q_{\delta_2}(t_0, x_0) \cap (\partial\{u > 0\} - R) = \emptyset$, $Q_{\delta_2}(t_0, x_0) \cap \partial\{u > 0\}$ is a $\mathbf{C}^{1/2,1}$-surface). Let us for now choose δ_1 with respect to $\varepsilon = \frac{1}{4}$ and let us assume that $(t_0, x_0) = 0$ and that $\nu(t_0, x_0) = e_n$. Then $u(s, y', y_n) = 0$ in $Q'_{\delta_1/2}(0) \times \{-(\sqrt{3}/2)\,\delta_1 < y_n < -1/(2\sqrt{3})\,\delta_1\}$ and $u(s, y', y_n) > 0$ in $Q'_{\delta_1/2}(0) \times \{(\sqrt{3}/2)\,\delta_1 > y_n > 1/(2\sqrt{3})\,\delta_1\}$. So we obtain a dense subset D of $Q'_{\delta_1/2}(0)$ such that for each $(t, x') \in D$ there exists a ball $B \subset \{u(t, \cdot, \cdot) = 0\}$ touching the free boundary in a point $(t, x', d(x'))$. From the upper semicontinuity, Proposition 5.1(4), from Proposition 8.1 as well as Lemma 7.1 we infer the existence of $\delta_2 \in (0, \delta_1/2)$ such that $E := \{(t, x', d(x')) \in \overline{Q_{\delta_2}(0)} : (t, x') \in D\} \subset R$. It follows that the cone-flatness (10) holds uniformly in $(\bar{t}, \bar{x}) \in E$ and implies that $\bar{E}$ is the graph of a $\mathbf{C}^{1/2,1}(\overline{Q'_{\delta_2}(0)})$-function g and that $\partial\{u > 0\} \cap \overline{Q_{\delta_2}(0)} = \text{graph } g$. Applying (10) once more with respect to arbitrary $\varepsilon > 0$ we see that g is Fréchet-differentiable with respect to the space variable.

Step 3 (Hölder-continuity of the normal $\nu(t, x)$). By Step 2 and Theorem 7.1

$$\int_{\partial B_1(0)} \left| \frac{u(t_1, x_1 + rx)}{r^{2/(1-\gamma)}} - \left(\frac{1-\gamma}{2} \max(x \cdot \nu(t_1, x_1), 0)\right)^{2/(1-\gamma)} \right| d\mathcal{H}^{n-1}$$
$$\leqslant C r^{\Lambda/2} \text{ for every } (t_1, x_1) \in \partial\{u > 0\} \cap \overline{Q_{\delta_2}(t_0, x_0)}$$
$$\text{and for every } r \leqslant \min(\delta_2, r_1).$$

We conclude that

$$\int_{\partial B_1(0)} \left| \left(\frac{1-\gamma}{2} \max(x \cdot \nu(t_1, x_1), 0)\right)^{2/(1-\gamma)} - \left(\frac{1-\gamma}{2} \max(x \cdot \nu(t_2, x_2), 0)\right)^{2/(1-\gamma)} \right| d\mathcal{H}^{n-1}$$
$$\leqslant C_1 r^{\Lambda/2} + \int_{\partial B_1(0)} r^{-2/(1-\gamma)} \int_0^1 |\nabla_{t,x} u(t_1 + s(t_2 - t_1), x_1 + rx + s(x_2 - x_1)) \cdot (t_2 - t_1, x_2 - x_1)| \, ds \, d\mathcal{H}^{n-1}$$
$$\leqslant C_1 r^{\Lambda/2} + C_2 r^{-2/(1-\gamma)} (|t_2 - t_1| \max(|t_2 - t_1|, |x_2 - x_1|, r)^{2\gamma/(1-\gamma) + \beta} + |x_2 - x_1| \max(|t_2 - t_1|, |x_2 - x_1|, r)^{(1+\gamma)/(1-\gamma)})$$
$$\leqslant (C_1 + C_2)(|x_1 - x_2|^{\sigma\Lambda/2} + |t_1 - t_2|^{\tau\Lambda/2})$$

where $\sigma = (1 + \Lambda/2)^{-1}$, $\tau = (2 - \beta + \Lambda/2)^{-1}$, $r := \max(|x_1 - x_2|^\sigma, |t_1 - t_2|^\tau) \leqslant \min(\delta_2, r_1)$ and β is the Hölder exponent obtained in Lemma 4.2.

Furthermore the left-hand side

$$\int_{\partial B_1(0)} \left| \left(\frac{1-\gamma}{2} \max(x \cdot \nu(t_1, x_1), 0)\right)^{2/(1-\gamma)} - \left(\frac{1-\gamma}{2} \max(x \cdot \nu(t_2, x_2), 0)\right)^{2/(1-\gamma)} \right| d\mathcal{H}^{n-1}$$
$$\geqslant c(n, \gamma) \, |\nu(t_1, x_1) - \nu(t_2, x_2)|$$

as can be easily shown by an indirect argument.

Corollary 19. *Let u be a solution of the Cauchy problem* (7). *Then* $((0, \infty) \times \mathbf{R}^n) \cap \partial\{u > 0\}$ *is the disjoint union of* R *and* Σ, $\partial\{u > 0\}$ *is locally* *in an open neighborhood of R a $\mathbf{C}^{1/2,\,1+\mu}$-surface and the set Σ is ignored by spatial integration by parts in $\{u>0\}$, i.e.*

$$\int_{\{u(t)>0\}} \partial_i \zeta = \int_{\partial_{\text{red}}\{u(t)>0\}} \zeta \nu_i \, d\mathscr{H}^{n-1} = \int_{R\cap\{s=t\}} \zeta \nu_i \, d\mathscr{H}^{n-1}$$

for a.e. $t\in(0,\infty)$ and every $\zeta\in C_0^{0,1}(\mathbf{R}^n)$; here the reduced boundary $\partial_{\text{red}}\{u(t)>0\}$ is defined as the set of free boundary points at which the outward normal of H. Federer [7, 4.5.5] exists.

Proof. Once it is proved that $\partial_{\text{red}}\{u(t)>0\}\subset R\cap\{s=t\}$, the result follows from Theorem 8.1 as well as from [5, Remark 7.1].

To obtain this inclusion, we consider a limit v_0 of the sequence $u(t, x_0+\rho_m x)/\rho_m{}^{2/(1-\gamma)} =: v_m(x)$ where $x_0\in\partial_{\text{red}}\{u(t)>0\}$. From Proposition 4.1, Theorem 5.2 and Proposition 4.2 we know that v_0 is a non-degenerate homogeneous solution of degree $2/(1-\gamma)$ which is (after rotation) $=0$ in $\{x_n\geqslant 0\}$ and positive a.e. in $\{x_n<0\}$. From [5, Lemma 7.1] we gather furthermore that $|\nabla v_0|^2\leqslant v_0^{1+\gamma}$ in $\mathbf{R}^n$ which implies $|v_0^{(1-\gamma)/2}(x)-v_0^{(1-\gamma)/2}(y)|\leqslant(1-\gamma)/2\,|x-y|$ for $x, y\in\mathbf{R}^n$. A short calculation yields now that

$$\Delta(v_0^{(1-\gamma)/2}) = \frac{1+\gamma}{1-\gamma}\, v_0^{-(1-\gamma)/2}\left(\left(\frac{1-\gamma}{2}\right)^2 - |\nabla(v_0^{(1-\gamma)/2})|^2\right) \text{ in } \{v_0>0\}$$

whereupon an application of [1, Lemma 4.6] to the function $v_0^{(1-\gamma)/2}$ leads to the conclusion that $v_0^{(1-\gamma)/2}(x)=(1-\gamma)/2\,\max(-x_n, 0)$ in $\mathbf{R}^n$ and thereby $(t, x_0)\in R\cap\{s=t\}$.

ACKNOWLEDGMENT

The author wishes to thank H. Matano for helping to clarify the dynamical systems aspect of this result.

Note added in proof. Another consequence of Corollary 8.1 is that for a.e. $t\in(0,\infty)$ and $\mathscr{H}^{n-1}$-a.e. $x\in\Sigma\cap\{s=t\}$ the spatial Lebesgue density of the set $\{u(t)>0\}$ must be either 0 or 1.

REFERENCES

1. H. W. Alt and D. Phillips, A free boundary problem for semilinear elliptic equations, *J. Reine Angew. Math.* **368** (1986), 63–107.
2. H. Amann, "Ordinary Differential Equations," Walter de Gruyter, Berlin/New York, 1990.
3. C. Bandle and I. Stakgold, The formation of the dead core in parabolic reaction-diffusion problems, *Trans. Amer. Math. Soc.* **286** (1984), 275–293.
4. L. A. Caffarelli, The regularity of free boundaries in higher dimensions, *Acta Math.* **139** (1978), 155–184.

5. H.-J. Choe and G. S. Weiss, A semilinear parabolic equation with free boundary, preprint, submitted (1997).
6. X.-Y. Chen, H. Matano, and M. Mimura, Finite-point extinction and continuity of interfaces in a nonlinear diffusion equation with strong absorption, *J. Reine Angew. Math.* **459** (1995), 1–36.
7. H. Federer, "Geometric Measure Theory," Springer, Berlin/Heidelberg/New York/Tokyo, 1969.
8. W. H. Fleming, On the oriented Plateau problem, *Rend. Circ. Mat. Palermo (2)* **11** (1962), 69–90.
9. A. Friedman and M. A. Herrero, Extinction properties of semilinear heat equations with strong absorption, *J. Math. Anal. Appl.* **124** (1987), 167–175.
10. A. Friedman and D. Kinderlehrer, A one phase Stefan problem, *Indiana Univ. Math. J.* **24** (1975), 1005–1035.
11. J. K. Hale, Topics in dynamic bifurcation theory, *in* "NFS-CBMS Lectures," Amer. Math. Soc., Providence, RI, 1981.
12. M. A. Herrero and J. J. L. Velazquez, On the dynamics of a semilinear heat equation with strong absorption, *Comm. Partial Differential Equations* **14** (1989), 1653–1715.
13. M. A. Herrero and J. J. L. Velazquez, Approaching an extinction point in one-dimensional semilinear heat equations with strong absorption, *J. Math. Anal. Appl.* **170** (1992), 353–381.
14. O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, "Linear and Quasi-linear Equations of Parabolic Type," Translations Mathematical Monographs, Vol. 23, Amer. Math. Soc., Providence, RI, 1988.
15. L. K. Martinson, The finite velocity of propagation of thermal perturbations in media with constant thermal conductivity, *U.S.S.R. Comput. Math. and Math. Phys.* **16** (1976), 141–149.
16. E. R. Reifenberg, An epiperimetric inequality related to the analyticity of minimal surfaces, *Ann. Math.* **80** (1964), 1–14.
17. Ph. Rosenau and S. Kamin, Thermal waves in an absorbing and convecting medium, *Physica D* **8** (1983), 273–283.
18. D. G. Schaeffer, Some examples of singularities in a free boundary, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **4** (1977), 133–144.
19. J. E. Taylor, Regularity of the singular sets of two-dimensional area-minimizing flat chains modulo 3 in $\mathbf{R}^3$, *Invent. Math.* **22** (1973), 119–159.
20. G. S. Weiss, Partial regularity for weak solutions of an elliptic free boundary problem, *Commun. Partial Differ. Equations* **23** (1998), 439–457.
21. G. S. Weiss, Self-similar blow-up and Hausdorff-dimension estimates for a class of parabolic free boundary problems, *SIAM J. Math. Anal.* **30** (1999), 623–644.